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Modeling Linear Quadratic Regulator LQR/LQT/LQGT for Inverted Pendulum System

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ABSTRACT

This paper attempts to apply lagrangian modeling and obtain inverted pendulum non-linear model and besides linearization of the obtained model around system work points, analyze some issues as stability, controllability and Observability. Later for linear model of system, it evaluates, designs and simulates optimal LQR regulators (linear quadratic regulator), LQT (linear quadratic tracker) and LQGT (linear quadratic Gaussian team). Finally, it concludes that we can perform states regulation and reference input tracking well for inverted pendulum non-linear system as even due to disturbances, system response is weakened and is eliminated in steady state. When the environment is noisy, required input tracking is done optimally as besides stabilization of system, the main output as one of the state variables tracks reference input optimally.
KEYWORDS: Inverted pendulum, Lagrangian modeling, LQR regulator, LQT tracker, LQGT tracker.

1. INTRODUCTION

Inverted pendulum is one of the classic system sin dynamic and control as recognized by some properties as non-linearity and inherent instability as one of the important issues in control engineering. The system is composed of a pendulum attached to a rotary arm rotating by the force imposed by DC motor. Thus, system input is DC motor feed voltage and the main output of system is angular situation of the arm attached to pendulum. In this system by applying suitable input, the arm shaft angle is changed as the pendulum angle is converged to zero (stable). Indeed, we try to create a state regulator by state feedback. As inverted pendulum is an unstable system, even with the smallest disturbance on system, pendulum is fallen. Thus, a regulator should be designed that besides system stabilization can reduce disturbance effect and eliminates it in steady state. The regulator for this purpose is optimal LQR of linear quadratic regulator making system states stable and weakening it and it eliminates disturbance effect in system response in case of steady state. Now, we want the system output tracks the inference input as the pendulum angle is stabilized and we try to design path tracker. We can use LQT as besides tracking reference input path can reduce disturbance effect and eliminates in steady state. If the environmental conditions are noise-based, LQGT tracker is used as weakening and eliminating the disturbance effect in steady state [1, 2].

2. Lagrangian modeling of inverted pendulum system

The studied system is experimental sample of inverted pendulum made in Quanser Company as shown in Figure 1.

Figure 1. Inverted pendulum system made in Quanser Company

The driving model for DC motor system is shown as in Figure 2.

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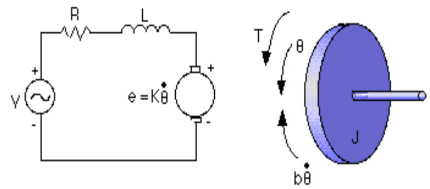

Figure 2. Modeling inverted pendulum system motor

$$V_m - R_m I_m - L_m \frac{dI_m}{dt} - V_{emf} = 0 \tag{1}$$

Back-emf voltage of motor is generated torsion by motor and output torsion from system as equations 2 to 4:

$$V_{emf} = K_m \omega_m = K_m \dot{\theta}_m \tag{2}$$

$$T_m = \eta_m K_t \left(\frac{V_m - V_{emf}}{R_m} \right) = \eta_m K_t \left(\frac{V_m - K_m \dot{\theta}_m}{R_m} \right) \tag{3}$$

$$T_{out} = \frac{\eta_m \eta_g K_t K_g (V_m - K_m K_g \dot{\theta})}{R_m} \tag{4}$$

Where,

r : Length of arm attached to pendulum

α : Pendulum angle situation

θ : Motor shaft

L : Distance of center of mass of pendulum

g : Constant gravity

m : Mass of arm attached to pendulum

J_{eq} : Inertia moment around θ

J_{Pcm}: Inertia moment around center of mass

For the above parameters, the following values are mentioned in quanser company as:

J_{eq}=0.0035842 & K_g=70 & K_m=0.00767

K_t=0.00767 & L=0.1675 & R_m=2.6 & g=9.81

m=0.125 & η_g=0.9 & η_m=0.69 & r=0.215

By determining the following state variables and linearization of model around its equilibrium points, the linearized model of system is shown as equation 8[3, 4]:

$$x_1 = \theta \ \& \ x_2 = \alpha \ \& \ x_3 = \dot{\theta} \ \& \ x_4 = \dot{\alpha} \tag{7}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 39.32 & -14.52 & 0 \\ 0 & 81.78 & -13.98 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 25.54 \\ 24.59 \end{bmatrix} v_m$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{8}$$

3. The evaluation of stability, controllability and Observability

Eigen values of system are obtained by Eig command in Matlab software as:

$\lambda_1 = 0$

$\lambda_2 = -17.1209$

$\lambda_3 = 7.5407$

$\lambda_4 = -4.9398$

As λ_3 eigen value is in the right side of dummy axis, the system is instable.

To evaluate system controllability by ctrb command in Matlab software, at first controllability matric is formed and then by rank command, controllability 4 matrix rank is obtained and the system is state controllable.

To evaluate Observability, by obsv command in Matlab software, besides obtaining observation matric by rank command, Observability matrix rank is 3 and the system is not state Observability system and has 1 unobserved mode [1].

1- Linear quadratic regulator (LQR)

One of the important and basic issues in linear optimal control is designing and implementation of Linear Quadratic Regulator (LQR) and this regulator aims to regulate and control linear system state variables about zero by taking optimal control costs in finite horizon or infinite horizon.

Generally, a linear time invariant system (LTI) is expressed by state space equations as:

$$\dot{x}(t) = Ax(t) + Bu(t)\ ,\ y(t) = Cx(t) + Du(t) \tag{9}$$

The criterion function that should be minimized is as:

$$J = \frac{1}{2}x^T(t_f)H(t_f)x(t_f) +$$
$$\frac{1}{2}\int_{t_0}^{t_f}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt$$

Where, Q is weight matrix of system states, R input weight matrix and H weight matrix of final cost. These components show relative importance of each of them and the higher the value of a parameter, the task has high importance compared to other goals. It is assumed that state variables x(t) and u(T) are not restricted, t_f is known and $x(t_f)$ is free and the optimal control minimizing J cost function is obtained as:

$$u^*(t) = -R^{-1}B^TP(t)x^*(t) = -K(t)x^*(t) \tag{11}$$

Where, K(t) matrix is called Kalman value and P(T) is definite positive symmetrical matrix (for all $t \in \left[t_0, t_f\right]$) obtained by solving Riccati differential equation:

$$\dot{P}(t) = -A^TP(t) - P(t)A + P(t)BR^{-1}B^TP(t) - Q$$

By final condition:

$$P(t = t_f) = H(t_f) \tag{13}$$

P(t) matrix is obtained by solving Riccati differential equation (12) has one steady state and transient part in Figure 3 as:

$$P(t = t_f) = H(t_f) = 0$$

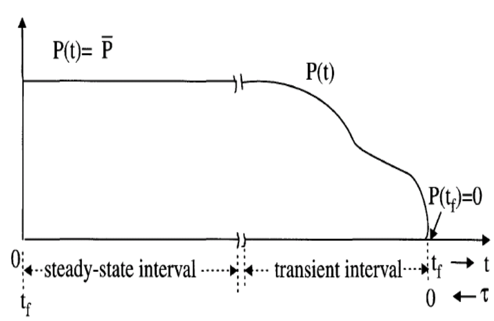

Figure 3. P(t) matrix obtained by equation 12

As shown in Figure 3, steady state P(T) has dedicated the majority and it continues to final time tf as matrix P(T) has two parts of steady state and transient and solving Riccati differential equation (12) is very difficult and to simplify the calculations, final time approaches infinite ($t_f \to \infty$), this causes the stead state P(T) goes to infinite and p(t) is considered with high approximation as $\bar{P}$ and LQR regulator is turned into a linear quadratic regulator with infinite horizon. As in infinite horizon, adding final cost function is not meaningful and the term of final cost including $H(t_f)$ is not in criterion function (10) and cost function is turned into the following form:

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt \tag{14}$$

During using infinite horizon, we should consider open loop system (9) is fully controllable.
Close loop optimal control and relevant Riccati equation are as:

$$u^*(t) = -R^{-1}B^T\overline{P}x^*(t) = -Kx^*(t) \tag{15}$$

$$-A^T\overline{P} - \overline{P}A + \overline{P}BR^{-1}B^T\overline{P} - Q = 0 \tag{16}$$

As the system is state controllable, LQR of infinite horizon of Q or weight matrices of system states can be considered as:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

lqr command is used in Matlab software of K matric or optimal value matrix for 3 different values R=0.1,1,10 for the main output of system and simulation is as Figure 4 (initial condition is zero):

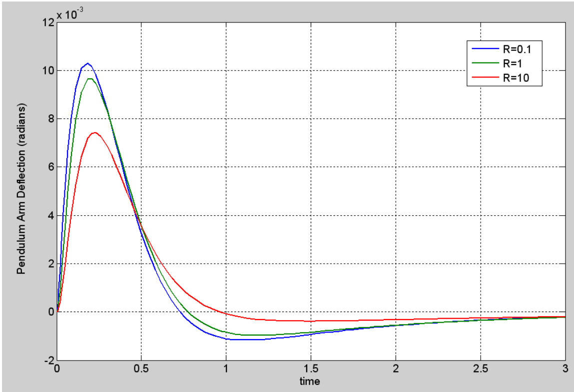

Figure 4. The behavior of motor shaft for 3 various values of R

As shown in the result of simulation, if R or input weight matrix is increased, it means that control energy importance is increased and the cost is decreased and this changes Q or state weight matrix of system for R=1 and shows its impact on system response performance as Figure 5.

$$Q_1 = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

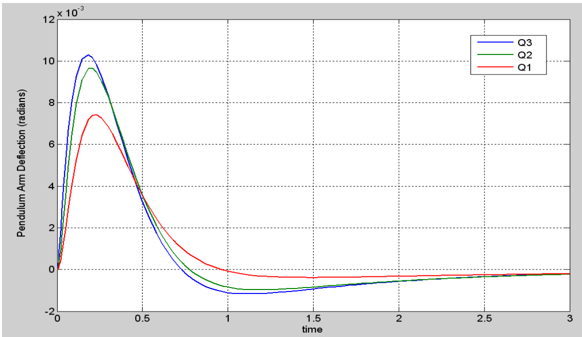

Figure 5. The behavior of motor shaft situation for 3 different values Q

Based on the result of simulation in Figure 6, increase of Q or weight matrix of system states increases system response zero approach and this means high costs. Indeed, optimal control signal range is increased by increasing Q value [2,5].

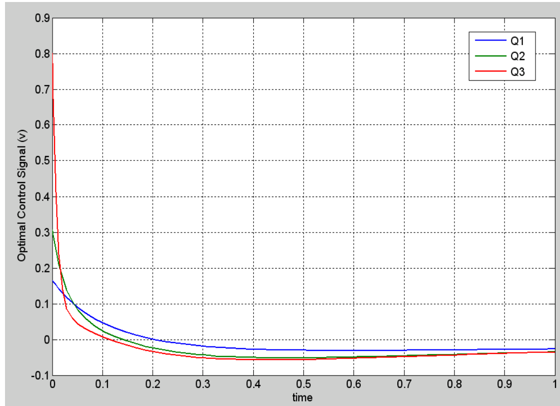

Figure 6. Optimal control signal $u^*(t)$ for three values of Q

4. Linear Quadratic Tracking LQT

In tracking systems, a system output tracks a favorite path as optimal. Indeed, this is generalizing regulator system as tracking error vector enters cost function to make system state variables stable (converging to zero) and system output tracks reference input and this tracking is due to convergence of new state variable of tracking error to zero.

For a linear time invariant (LTI) system, cost function is defined for LQT tracking for following finite horizon as:

$$J = \frac{1}{2}\{x^T(t_f)H(t_f)x(t_f) + e^T(t_f)F(t_f)e(t_f)\} +$$

$$\frac{1}{2}\int_{t_0}^{t_f}\left[x^T(t)Q_x x(t) + e^T Q_e e + u^T(t)Ru(t)\right]dt \tag{17}$$

Where Q_x is weight matrix of system states, Q_e is tracking error weight matrix (e), R is input weight and H,F are weight matrix of error of tracking and system states in final time. Equation (17) is written as:

$$J = \frac{1}{2}\xi^T(t_f)G(t_f)\xi(t_f) +$$

$$\frac{1}{2}\int_{t_0}^{t_f}\left[\xi^T(t)Q_\xi\xi(t) + u^T(t)Ru(t)\right]dt$$

As:

$$\xi(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \quad , \quad G = \begin{bmatrix} H & 0 \\ 0 & F \end{bmatrix}$$

$$Q_\xi = \begin{bmatrix} Q_x & 0 \\ 0 & Q_e \end{bmatrix} \tag{19}$$

In this section, it is assumed that $\xi(t)$ and u(t) state and control variables are not restricted, t_f is known and $\xi(t_f)$ is free and the optimal control minimizing J cost function is as:

$$u^*(t) = -R^{-1}B^TP(t)\xi^*(t) = -K(t)\xi^*(t) \qquad (20)$$

P(t) matrix is obtained by equation 12 and as the previous section, Riccati differential equation solution is very difficult due to steady state and transient sections in P(t) answer. To make the calculations simple, the final time approaches infinite to consider P(t) with high approximation as $\overline{P}$ and LQT tracking can be turned into a linear quadratic tracking with infinite horizon.

Like LQR section, adding final cost function is not meaningful in infinite horizon and the term of final costs including $G(t_f)$ is not in criterion function (18) and the cost function is turned into the following form:

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left[\xi^T(t)Q_\xi\xi(t) + u^T(t)Ru(t)\right]dt \tag{21}$$

During using infinite horizon, the system should be fully controllable and state controllability condition guarantees that optimal cost is finite. In this state, the relevant Riccati equation to equation 16 is determined and loop optimal control is obtained as:

$$u^*(t) = -R^{-1}B^T\overline{P}\xi^*(t) = -K\xi^*(t) \tag{22}$$

For LQT tracking system, open loop system state space with tracking error state variable is shown as equation 22:

$$\dot{\xi}(t) = \hat{A}\xi(t) + \hat{B}u(t)\,,\ y(t) = \hat{C}\xi(t) + \hat{D}u(t) \tag{23}$$

As

$$\hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad , \hat{A} = \begin{bmatrix} A & [0] \\ -C & 0 \end{bmatrix}$$

$$\hat{D} = D \text{ و } \hat{C} = \begin{bmatrix} C & 0 \end{bmatrix} \tag{24}$$

As the main output of system (angular situation of arm attached to pendulum) is equal to x_1 state variable and matrix C is considered as:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

The state space model of open loop system with tracking error state variable is as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 39.32 & -14.52 & 0 & 0 \\ 0 & 81.78 & -13.98 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ e \end{bmatrix} \quad , \quad \dot{e} = r - Cx$$

In this section, by applying reference input (as angle situation) load shaft tracks reference input by rotation and other states variables are converged to zero (stable). As state controllability condition is satisfied for system 24, LQT can be implemented for the above system as Q_ξ or weight matrix of system states is as:

$$Q_\xi = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}$$

By calculation of matrix K for 3 various values R=0.01,0.1,1 for main output of system (motor shaft situation) is applied and simulation result is obtained as Figure 7 by considering unified step reference input by 1 rad as Figure 7 (initial conditions 0.1):

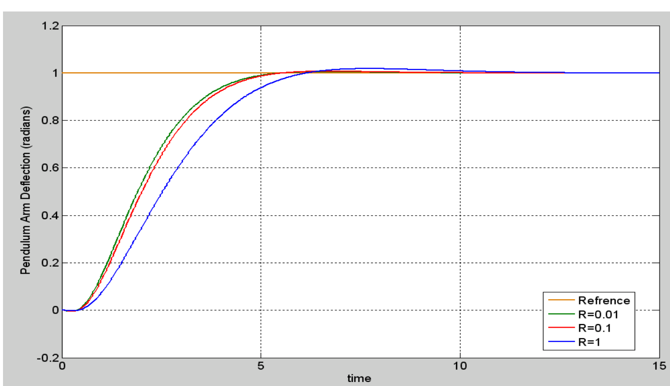

Figure 7. The behavior of tracking motor shaft situation for 3 various values of R

Optimal control signal $u^*(t)$ for various 3 values of R is as:

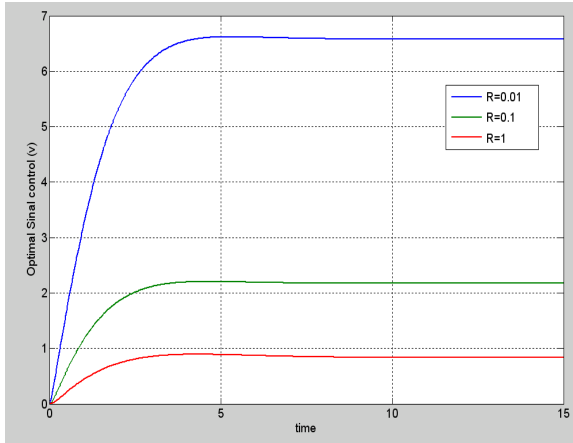

Figure 8. Optimal control signal $u^*(t)$ for three values of R

If R is increased, it means that control energy is of great importance than before and it is lower than control cost and this makes weak performance of system response and based on Figure 10, response speed is reduced by increasing R value.

For 3 values of 0.1,1,5 (for R=0.1), Q_e or weight matrix of error integral (e) is changed and its impact on input tracking is obtained as Figure 9:

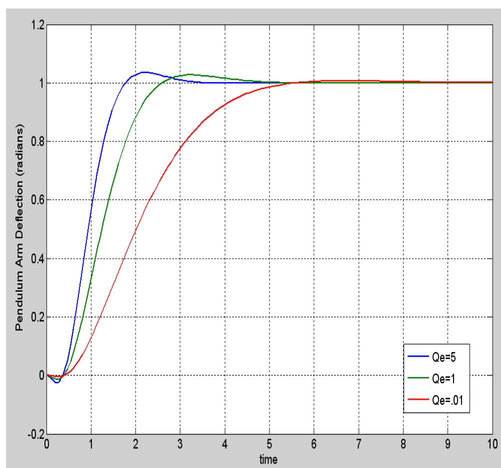

Figure 9. The behavior of tracking motor shaft for 3 different values of Q_e

If Q_e value or error weight matrix of Q_e is increased, output response error is of great importance than before and output response can approach reference input and response error is decreased.

To observe the input impact of unified stair disturbance on system output response, simulation result for Q_ξ of Figure 7 is three various values R=0.01,0.1,1, disturbance of unified stair in t-5sec and stair reference input with 5rad is as :

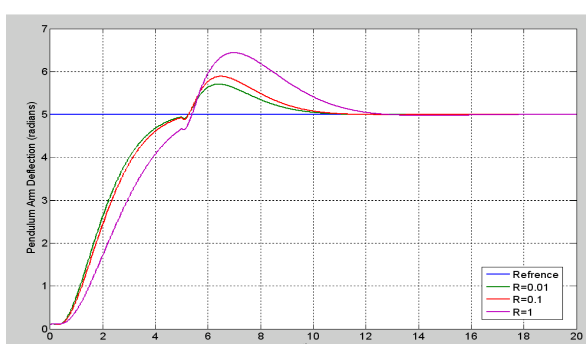

Figure 10. The behavior of tracking motor shaft in the presence of unified stair disturbance for 3 different values of R

As shown in Figure 10, LQT tracking can weaken the impact of fixed disturbance (unified stair) in response of close loop system and eliminates in steady state case and if R value is increased, it means that system entrance or control energy is of great importance than before and control cost is lowered and it means the weak performance of close loop system and reference input tracking is weaker [2, 5].

5. Linear Quadratic Gaussian Tracking

Instead of states, we use their estimation to track the required path and the combination of LQT tracker and Kalman filter (a generalized state observer) leads to a type of path tracking as Linear Quadratic Gaussian (LQG). This tracker can create state feedback by having noise outputs and this leads to the minimization of a performance function. Controller inputs include output feedback y and reference input r by which u control signal is generated and enters the main system with w process noise, a measurement noise v is added with output y. The schema of LQGT is shown in Figure 11:

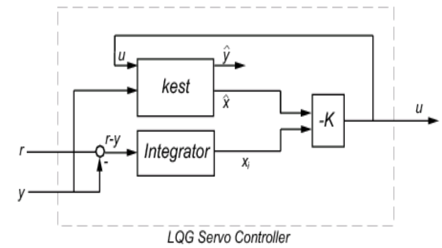

Figure 11. Tracker of LQGT path

As shown in Figure 11, controller includes Kalman filter, integral and LQI block. Output feedback and control signal u are used by Kalman filter to have optimal estimation of system states that later by estimation states and tracking error integral, K optimal value vector is generated by LQI block and is multiplied by estimated states vector to create u control signal.

As process and measurement noises enter close loop system, their model should be considered as equation 25 in state space equations of close loop system.

$$\dot{x} = Ax + Bu + Gw \,,\; y = Cx + Du + Hv \tag{25}$$

As in this controller, Kalman filter as a state observer is used, the main system should be state full observable and as the studied system is state non-observable, by defining output as:

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x$$

the system can be state observable and y1, y2 outputs are load shaft angular situation (arm attached to pendulum) and pendulum based on radian. In equation 24, v,w are static white Gaussian noise with zero mean indicated measurement noise and process, respectively and v,w are not dependent and their covariance matrix is as:

$$Q_{vv} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} \quad \& \quad Q_{ww} = 0.01$$

G,H are equal to:

$$G = \begin{bmatrix} 0 \\ 0 \\ 25.54 \\ 24.59 \end{bmatrix} \quad \& \quad H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For a linear time invariant system, cost function for LQGT tracking for infinite horizon is defined as:

$$J = \int_0^\infty (x^T Q_x x + x_i^T Q_{ei} x_i + u^T R u) dt \tag{26}$$

Where, Q_x is weight matrix of system states, Q_{ei} weight matrix of error integral (x_i) and R input weight matrix and like the previous section, these components show the importance of each of them to another component. Q matrix including Q_x, Q_{ei} matrices is formed as:

$$Q = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For three various values R=0.01,10,15 , simulation result is as Figure 12:

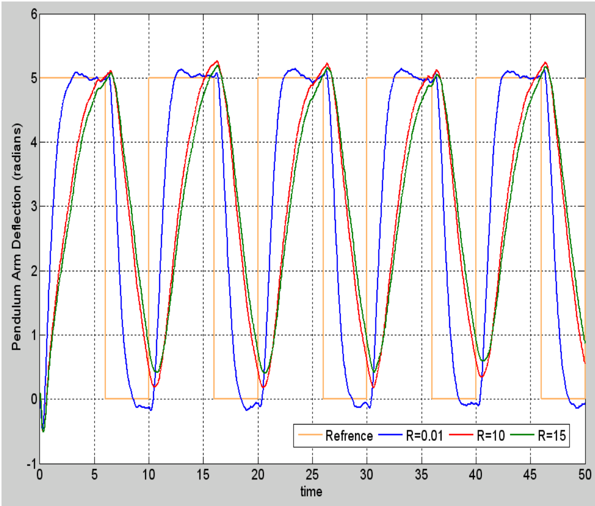

Figure 12. Tracker motor shaft situation behavior for 3 values of R

Optimal control signal $u^*(t)$ is as follows:

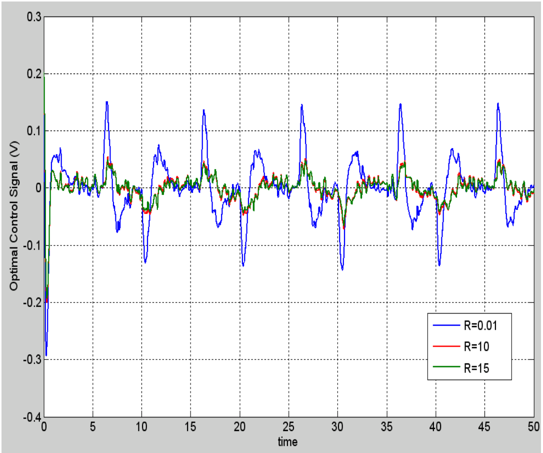

Figure 13. Optimal control signal $u^*(t)$ for three values of R

By increasing R, control energy is reduced and tracking is weak and control signal has steep trend at the beginning of load shaft condition change and it is due to the rapid settling of shaft to compensate pendulum angular condition in movement to have an optimal control besides reference input tracking and when R is increased, settling is reduced and control quality is reduced. For 3 different values of Q_{ei} =1, 1. 0, 12 and R=0.01, the following results are achieved:

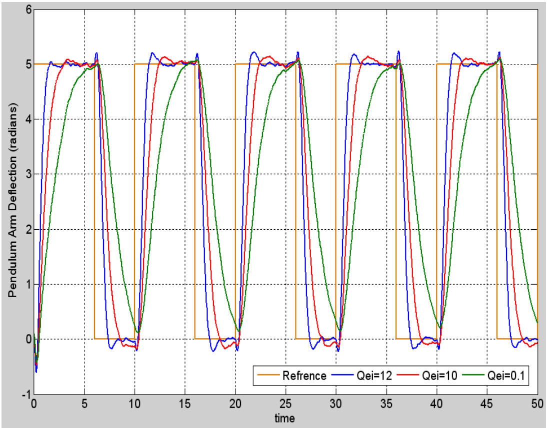

Figure 14. The tracking motor shaft situation for three values of Q_{ei}

As shown in Figure 14, if Q_{ei} is increased, output response can be closer to reference input and tracking can be done better and now for three values of Q_x=0. 1,0.15,0.2, Q_{ei}=1 and R=0.01 and the result of simulation is as:

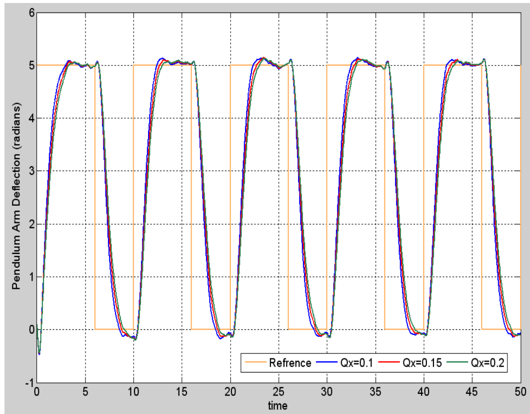

Figure 15. The behavior of tracking motor shaft for three values of Q_x

As shown in Figure 15, if Q_x is increased, estimation error can be higher and input tracking has high error. To investigate the impact of unified stair disturbance adding with the main system integrating input on output response, the result of simulation for three values Q_{ei}=1, 12, 150 , Q_x=0.1 and R=0.01 is as:

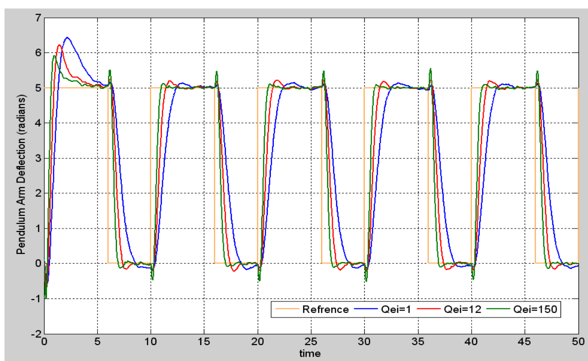

Figure 16. The response of tracking close loop in the presence of unified stair disturbance for three values of Q_{ei}

As shown in Figure 16, if Q_{ei} is increased, response overshooting can be reduced and response speed is increased and error is reduced. This makes load shaft settling and the system has limitation in this regard as to reduce overshooting, Q_{ei} value is increased considerably and load shaft should perform rapid and more settling and it damages the shaft. For example, for Q_{ei}=10000, the response of close loop system in presence of unified stair disturbance is shown in Figure 18.

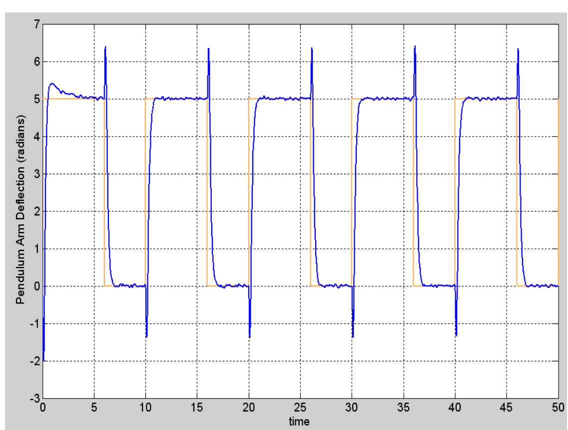

Figure 17. Tracker close loop system response in presence of unified stair disturbance for Q_{ei}=10000.

As shown in Figure 17, overshooting is reduced very much but load shaft sudden settling is increased, thus we cannot increase Q_{ei} very much. As shown in Figure 17, the response of close loop system in the presence of unified stair disturbance of Q_{ei} =150 has overshooting of about 16% and it is suitable and there is no need to increase Q_{ei} to achieve a suitable response [2, 6].

7. Conclusion

By lagragian modeling in this study, we achieved non-linear model of inverted pendulum system and after linearization of this model around system equilibrium point, we performed other analyses on this linearized model. It was shown that this system is instable, state controllable and stat non-observable. Then, we considered linear quadratic regulator (LQR) and it was shown that LQR regulator generated the best state feedback value to minimize production cost function and by minimizing it, the states were converged to zero (stable), then, we discussed linear quadratic tracking (LQT) and we observed that the principles of this tracker is generalizing LQR regulator as by entering tracking error as a new state variable in cost function, minimizing performance index guarantees zero reference input tracking error, otherwise the optimal cost is infinite. As we observed, LQT tracker weakens disturbance effect in system response and eliminates it in steady state and finally LQGT tracker was investigated as for reference input tracking, instead of using system states, their estimation by Kalman filter is used. In other words, LQGT is an optimized tracker of path in the presence of noise and we concluded that LQGT weakens disturbance effect in system response and eliminates it in steady state and we observed that to reduce response error considerably, control energy with high peaks should be during the change of load shaft to track reference input. Thus, high reduction of error has practical limitations and we can not do this and to have suitable performance of response, we don't need error much reduction.

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