New Results in \( Q \) -Inner Product

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ABSTRACT

In this paper, we establish some new results in \( (Q) \) – inner product. These are different from the results published in the book, “Semi-inner products and application” by S. S. Dragomir.

KEYWORDS: Inner product, Complex inner product, Q-inner product,

PRELIMINARIES

Dragomir (see, [1]-[4]) introduced some generalization of inner product in a real linear space that extends this concept in a different manner than the extension due to Lumer-Giles, Tapia or Miličić (see, [1]).

The following definitions are used by Dragomir in [1].

Definition: A mapping \((...,...,)\) \( : \mathbb{X}^4 \rightarrow \mathbb{R} \) will be called a quaternary-inner product, or \((Q)\) - inner product, for short, if the following conditions are satisfied:

\[(i) \quad (\alpha x_1 + \beta x_2, x_3, x_4, x_5) = \alpha (x_1, x_3, x_4, x_5) + \beta (x_2, x_3, x_4, x_5) \quad \text{where} \quad \alpha, \beta \in \mathbb{R} \quad \text{and} \quad x_i \in \mathbb{X} \quad \forall \ i = 1, 5.\]

\[(ii) \quad (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) = (x_1, x_2, x_3, x_4) \quad \text{for any} \quad \sigma \quad \text{a permutation of the indices} \quad (1, 2, 3, 4) \quad \text{and} \quad x_i \in \mathbb{X} \quad \forall \ i = 1, 4.\]

\[(iii) \quad \text{One has the following Schwartz type inequality} \]

\[
\left\| (x_1, x_2, x_3, x_4) \right\|_4 \leq \prod_{i=1}^{4} \left( x_i, x_i, x_i, x_i \right)_q,
\]

\[(1.1) \quad \text{for all} \quad x_i \in \mathbb{X}, \quad \forall \ i = 1, 4 \quad \text{and} \quad (x_i, x_i, x_i, x_i)_q > 0 \quad \text{if} \quad x_i \neq 0.\]

Definition A real linear space \( \mathbb{X} \) endowed with a \((Q)\) -inner product \((...,...,)\) on it will be called a \((Q)\) -inner product space. Now by the definition of \((Q)\) -inner product space, we can state the following simple properties:

\[
(0, x_2, x_3, x_4)_q = 0,
\]

and

\[
(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4)_q = \alpha^4 (x_1, x_2, x_3, x_4)_q,
\]

for any \( \alpha \in \mathbb{R} \) and \( \forall \ x_1, x_2, x_3, x_4 \in \mathbb{X}. \)

Dragomir [1] also pointed out proposition that followed by the definition of \( Q \) – inner product (see definition (01)) using two vectors.

Proposition: Let \((X, \left\| \cdot \right\|_q)\) be a \( Q \) – normed space. Then for all \( x_1, x_2 \in X \), we have

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\[ \|x_1 + x_2\|_q^4 + \|x_1 - x_2\|_q^4 = 2(\|x_1\|_q^4 + \|x_2\|_q^4) + 12(x_1, x_1, x_2, x_2)_q \]  
(1.2)

and

\[ \|x_1 + x_2\|_q^4 + \|x_1 - x_2\|_q^4 \leq 2(\|x_1\|_q^4 + \|x_2\|_q^4) + 12\|x_1\|_q^2\|x_2\|_q^2. \]  
(1.3)

**MAIN RESULTS**

Main object of this paper is to extend the idea of Dragomir [Drag1] given in equation (1.2) and inequality (1.3) from two vectors to four vectors.

**Proposition**  Let \((X, (\ldots, \ldots)_q)\) be a \((Q)\) -inner product space. Then the mapping 

\[ \|\cdot\|_q : X \to \mathbb{R}, \|x\|_q = (x, x, x, x)_q^{\frac{1}{4}} \]

is a norm on \(X\).

**Proof:** Using definitions 1 and 2 of inner product and by simple calculation

\[ \|x_1 + x_2 + x_3 + x_4\|_q^4 = \sum_{i=1}^{4} \|x_i\|_q^4 + 4 \sum_{i \neq j=1}^{4} (x_i, x_i, x_j, x_j)_q \]

\[ + 6 \sum_{i \neq j=1}^{4} (x_i, x_i, x_j, x_j)_q + 12 \sum_{i \neq j=1}^{4} (x_i, x_i, x_j, x_k)_q \]

\[ + \sum_{i \neq j \neq k \neq l=1}^{4} (x_i, x_i, x_j, x_l)_q, \]

\(\forall x_1, x_2, x_3, x_4 \in X.\)

From inequality (1.1), we have

\[ (x_i, x_i, x_i, x_j)_q \leq \|x_i\|_q^4 \|x_j\|_q, \]  
(2.2)

\[ (x_i, x_i, x_j, x_j)_q \leq \|x_i\|_q^2 \|x_j\|_q^2, \]  
(2.3)

\[ (x_i, x_i, x_j, x_k)_q \leq \|x_i\|_q^2 \|x_j\|_q \|x_k\|_q, \]  
(2.4)

Using the eq.(2.2)-(2.4), equation (2.1) becomes

\[ \|x_1 + x_2 + x_3 + x_4\|_q^4 \leq \sum_{i=1}^{4} \|x_i\|_q^4 + 4 \sum_{i \neq j=1}^{4} (\|x_i\|_q^2 \|x_j\|_q^2) \]

\[ + 6 \sum_{i \neq j=1}^{4} (\|x_i\|_q^2 \|x_j\|_q^2) \]

\[ + 12 \sum_{i \neq j \neq k \neq l=1}^{4} (\|x_i\|_q^2 \|x_j\|_q \|x_k\|_q \|x_l\|_q) \]

\[ + \sum_{i \neq j \neq k \neq l=1}^{4} (\|x_i\|_q^2 \|x_j\|_q \|x_k\|_q \|x_l\|_q), \]

i.e.
which produces the inequality

$$\|x_1 + x_2 + x_3 + x_4\|_q \leq \left( \sum_{i=1}^{4} \|x_i\|_q \right)^4,$$

on the other hand, we have

$$\|x_i\|_q \geq 0 \forall x_i, i = 1, 4 \in X,$$

and

$$\|x_i\|_q = 0 \Rightarrow x_i = 0,$$

and finally, we also have:

$$\|\alpha x_i\|_q = |\alpha| \|x_i\|_q,$$

where $\alpha \in \mathbb{R}$ and $x_i \in X$.

Consequently $\|\|_q$ is a norm and the proposition is proved.

**Proposition** Let $(X, (\cdot, \cdot, \cdot, \cdot)_q)$ be a $\mathbb{Q}$-normed space. Then for all $x_1, x_2, x_3, x_4 \in X$, we have:

$$\|x_1 + x_2 + x_3 + x_4\|_q \leq \left( \sum_{i=1}^{4} \|x_i\|_q \right)^4 \leq \sum_{i=1}^{4} \|x_i\|_q.$$

(2.6)

**Proof**: We know from (2.1) that

$$\|x_1 + x_2 + x_3 + x_4\|_q^4 = \sum_{i=1}^{4} \|x_i\|_q^4 + 4 \sum_{i < j=1}^{4} \left( x_i, x_j, x_i, x_j \right)_q$$

$$+ 6 \sum_{i < j < k=1}^{4} \left( x_i, x_j, x_k, x_j \right)_q + 12 \sum_{i < j < k < l=1}^{4} \left( x_i, x_j, x_k, x_l \right)_q$$

Similarly computing other seven terms in (2.6) and adding we obtain the required result.

**Proposition**: Let $(X, (\cdot, \cdot, \cdot, \cdot)_q)$ be a $\mathbb{Q}$-normed space. Then for all $x_1, x_2, x_3, x_4 \in X$, we have:
\[ \|x_1 + x_2 + x_3 + x_4\|_q^4 + \|x_1 + x_2 - x_3 - x_4\|_q^4 + \|x_1 - x_2 + x_3 + x_4\|_q^4 \]
\[ + \|x_1 - x_2 - x_3 + x_4\|_q^4 + \|x_1 + x_2 + x_3 - x_4\|_q^4 + \|x_1 + x_2 - x_3 + x_4\|_q^4 \]
\[ + \|x_1 - x_2 + x_3 + x_4\|_q^4 + \|x_4 + x_2 + x_3 - x_1\|_q^4 \leq 8 \sum_{i=1}^{4} \|x_i\|_q^4 + 48 \sum_{i \neq j=1}^{4} \left( \|x_i\|_q^2 \|x_j\|_q^2 \right) \]  

(2.7)

Proof: From proposition 05 and using following inequalities
\[ (x_1, x_1, x_2, x_2)_q \leq \|x_1\|_q^2 \|x_2\|_q^2, \]
\[ (x_1, x_1, x_3, x_3)_q \leq \|x_1\|_q^2 \|x_3\|_q^2, \]
\[ (x_1, x_1, x_4, x_4)_q \leq \|x_1\|_q^2 \|x_4\|_q^2, \]
\[ (x_2, x_2, x_3, x_3)_q \leq \|x_2\|_q^2 \|x_3\|_q^2, \]
\[ (x_2, x_2, x_4, x_4)_q \leq \|x_2\|_q^2 \|x_4\|_q^2, \]
\[ (x_3, x_3, x_4, x_4)_q \leq \|x_3\|_q^2 \|x_4\|_q^2, \]
we obtain the required (2.7).

**REFERENCES**

[1] S. S. Dragomir, Semi-inner products and application, School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.


