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The exponentiated Topp-Leone distribution: Properties and application

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ABSTRACT

The Topp-Leone distribution was first introduced by Topp and Leone (1955) as a probability distribution with bounded support which is useful for modelling life-time phenomena. In this paper, we propose a generalization of the Topp-Leone distribution referred to as the exponentiated Topp-Leone distribution. We study many aspects of the new model like hazard rate function, the moments and the order statistics. We discuss maximum likelihood estimation of the model parameters. In application to a real data set, we show that the exponentiated Topp-Leone model can be used quite effectively in analyzing data.
KEYWORDS: Hazard Function, Maximum Likelihood , Topp-Leone Distribution.

1. INTRODUCTION

The two-parameter Topp-Leone [1] distribution is useful for modelling bounded lifetime phenomena and different aspects of this distribution have been studied by Nadarajah and Kotz [2]. We denote the Topp-Leone distribution with positive parameters ν and b by TL (ν, b). If X~TL (ν, b), then its cumulative distribution function (cdf) is:

$$G(x)=\left(\frac{x}{b}\right)^{\nu}\left(2-\frac{x}{b}\right)^{\nu}, \qquad 0<x<b, \tag{1}$$

And the corresponding probability density function (pdf) is

$$g(x)=\frac{2\nu}{b}\left(1-\frac{x}{b}\right)\left(\frac{x}{b}\right)^{\nu-1}\left(2-\frac{x}{b}\right)^{\nu-1}, \qquad 0<x<b. \tag{2}$$

In this paper, we aim at introducing a new extension of the Topp-Leone distribution by using the exponentiation method. A random variable X is said to have the exponentiated Topp-Leone (ETL) distribution if its pdf and cdf are, respectively, given by

$$f(x)=\frac{2\nu\beta}{b}\left(1-\frac{x}{b}\right)\left(\frac{x}{b}(2-\frac{x}{b})\right)^{\nu-1}\left(1-\left(\frac{x}{b}\right)^{\nu}\left(2-\frac{x}{b}\right)^{\nu}\right)^{\beta-1}, \; 0<x<b, \; \nu,\beta>0, \tag{3}$$

and

$$F(x)=1-\left(1-\left(\frac{x}{b}\right)^{\nu}\left(2-\frac{x}{b}\right)^{\nu}\right)^{\beta}, \qquad 0<x<b. \tag{4}$$

We write X~ETL (ν, b, β) if the pdf of X can be expressed as 3. In the sequel, we discuss several properties of this new model and provide an application in the end.

2. Mathematical Properties

2.1. Shape of Density

The following results can be deduced:

- If $0<\nu\le 1$, then this distribution is J-shaped (decreasing) because $f(x)>0$, $f'(x)<0$ and $f''(x)>0$ for all $0<x<b$, where $f'(x)$ and $f''(x)$ are, respectively, the first and second derivative of $f(x)$.
- If ν ≥ 1, β ≥ 1, and then the density function has a maximum point.
- If ν ≥ 1, β < 1, then the density function is increasing.

Note that the pdf of the Topp-Leone distribution can not be increasing while the ETL distribution can involve increasing pdfs. Therefore, we may conclude that the ETL distribution is more flexible in the sense that it can fit data sets more effectively in comparison with the Topp-Leone distribution. Figure 1 plots the density function for selected parameter values.

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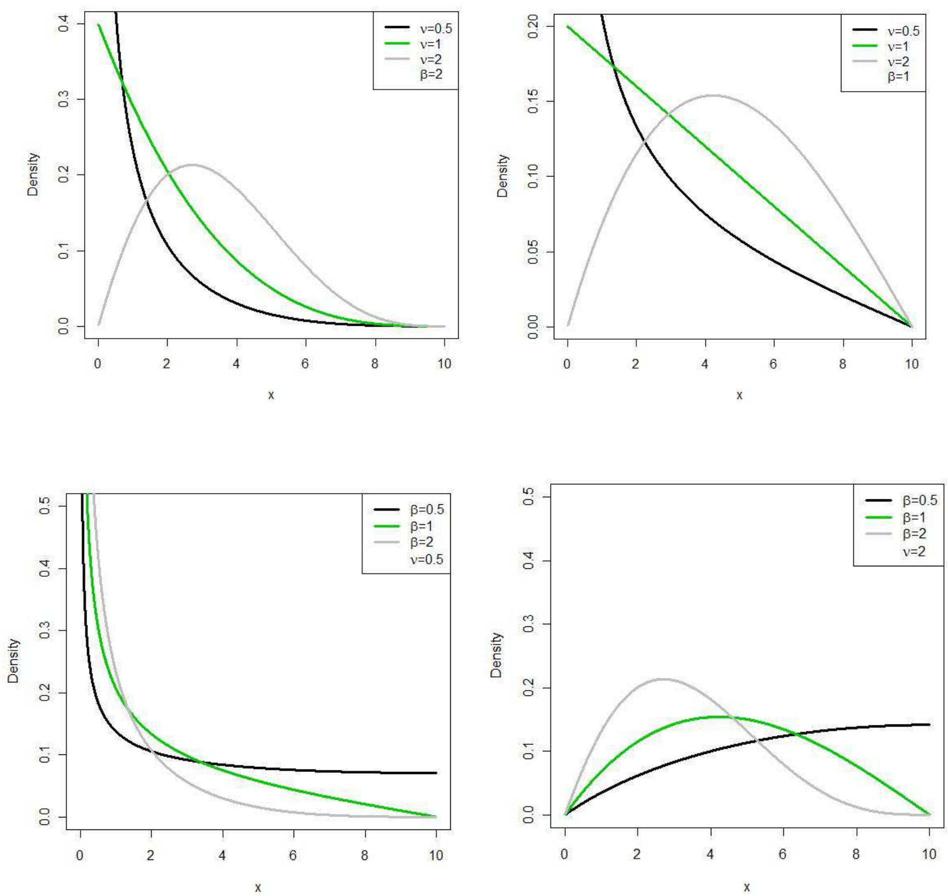

Fig 1. Pdfs of the ETL distribution for some selected parameter combinations when b=10

2.2. Expansions for the pdf and cdf

Using the series expansion,

$$(1-z)^a = \sum_{i=0}^{\infty} \binom{a}{i} (-z)^i$$

The cdf and the pdf of the ETL distribution can be expanded as

$$F(x) = 1 - \sum_{i=0}^{\infty} \binom{\beta}{i} (-1)^i \left[\left(\frac{x}{b} \right) \left(2 - \frac{x}{b} \right) \right]^{\nu i}.$$

and

$$f(x) = \frac{2\nu\beta}{b} \left(1 - \frac{x}{b}\right) \left[\left(\frac{x}{b} \right) \left(2 - \frac{x}{b} \right) \right]^{\nu - 1} \sum_{i=0}^{\infty} \binom{\beta - 1}{i} (-1)^i \left[\left(\frac{x}{b} \right)^{\nu} \left(2 - \frac{x}{b} \right)^{\nu} \right]^i \quad (5$$

$$= \frac{2\nu\beta}{b} \left(1 - \frac{x}{b}\right) \sum_{i=0}^{\infty} \binom{\beta - 1}{i} (-1)^i \left[\left(\frac{x}{b} \right) \left(2 - \frac{x}{b} \right) \right]^{\nu(i+1) - 1},$$

respectively.

2.3. Hazard Rate Function

Let X be a non-negative continuous random variable, usually representing the time to failure of a unit or a system, with pdf $f(x)$ and cdf $F(x)$. It is well known that an important measure of ageing is the hazard rate (HR), defined as

$$\lambda(x) = \lim_{\Delta x \to 0} \frac{P(X < x + \Delta x | X > x)}{\Delta x} = \frac{f(x)}{\overline{F}(x)},$$

Where $\overline{F}(x) = 1 - F(x)$ is the survival function of *X*. That is, $\lambda(x)\Delta(x)$ represents the conditional probability that an individual of age *x* will fail whitin the interval ($x, x+\Delta x$), see Ghitany et al., [3]. For the Exponentiated Topp-Leone distribution, the HR is easily calculated to be

$$\lambda_{ETL}(x) = \frac{2\nu\beta}{b}\frac{(1-y^2)^{\nu-1}y}{1-(1-y^2)^{\nu}}, \qquad y = 1-\frac{x}{b},\ 0<x<b. \qquad (6$$

Comparing the HR function of ETL distribution with the one of the Topp-Leone distribution, denoted as $\lambda_{TL}(x)$, we conclude that

$$\lambda_{ETL}(x) = \beta\,\lambda_{TL}(x).$$

The following results can be deduced

- If $0<\nu<1$, Then $\lambda_{ETL}(x)$ takes the "bathtub" shape.
- If $\nu \geq 1$, then $\lambda_{ETL}(x)$ is a strictly increasing function.

Figure 2 plots $\lambda_{ETL}(x)$ for selected parameter values.

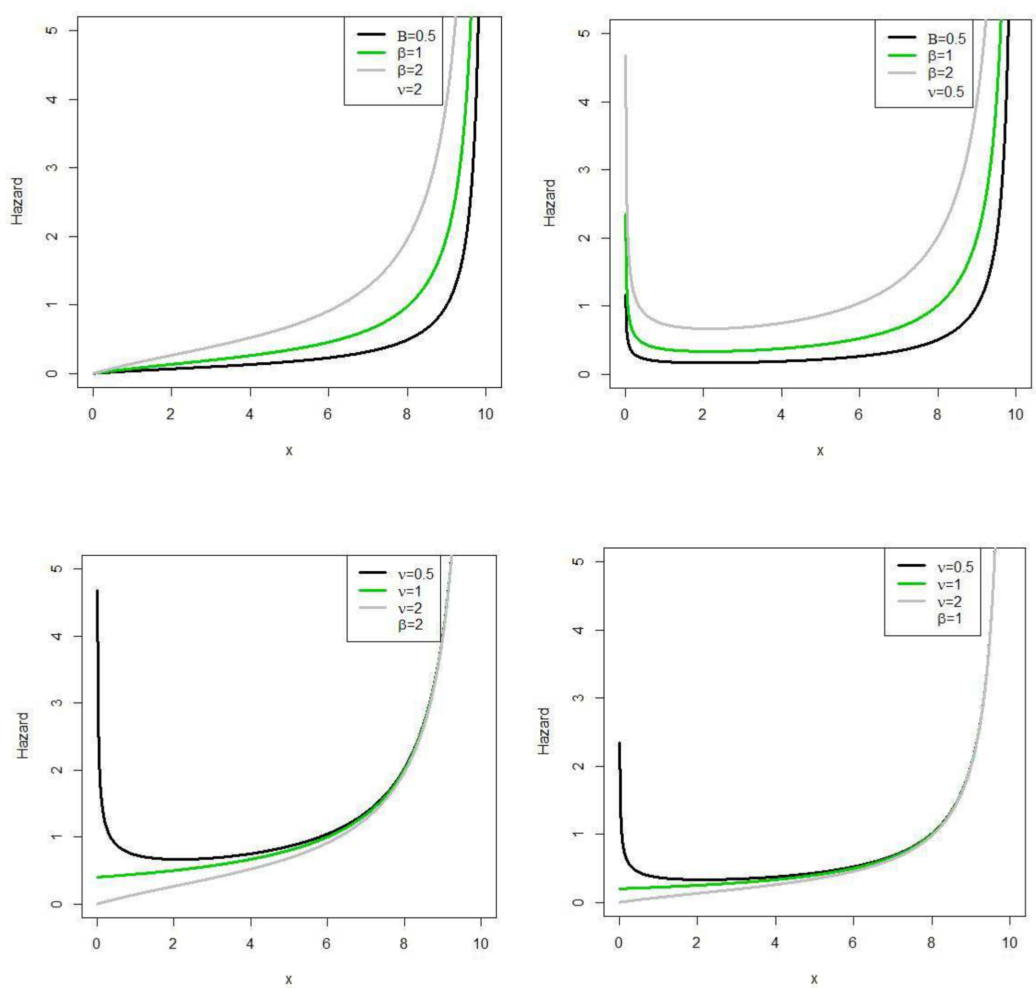

Fig 2. Hazard rate function of the ETL distribution for some selected parameter combinations when b=10

2.4. Mean Residual Lifetime

Given that a component survives up to time $t>0$, the residual life is the period beyond t until the failure time which is defined by the conditional random variable $X-t|X>t$. It is well-known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely, see Gupta and Gupta [4]. Here, we intend to derive the r^{th} order moment of the residual life of the ETL distribution using the following equation.

$$\mu_r(t) = E\left[(X-t)^r \middle| X > t\right] = \frac{1}{\overline{F}(t)}\int_t^b (x-t)^r f(x)dx$$ (7
$$= (-t)^r + \frac{1}{\overline{F}(t)}\sum_{k=1}^{r}\binom{r}{k}(-t)^{r-k}\int_t^b x^k f(x)dx.$$

Now we have

$$\int_t^b x^j f(x)dx = \frac{2\nu\beta}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\beta-1}{i}\binom{\nu(1+i)-1}{j}(-1)^{i+j}2^{\nu(1+i)-j-1}\int_t^b x^k(1-\frac{x}{b})(\frac{x}{b})^{\nu(1+i)+j-1}dx.$$

The reversed hazard rate (RHR) of a non-negative continuous random variable X with pdf and cdf at time x is defined as

$$a(x) = \lim_{\Delta x \to 0}\frac{P(X > x - \Delta x | X \le x)}{\Delta x} = \frac{f(x)}{F(x)}.$$

That is $a(x)\Delta x$ defines the conditional probability of a failure of a unit in $(x-\Delta x, x)$ given that the failure had occurred in $[0, x]$, see Ghitany et al., [3]. For the ETL distribution the RHR is the equation 3 divided by equation 4.

2.5. Moments

If X~ETL (ν, b, β), the kth moment of X can be derived as

$$E(X^k) = \int_0^b x^k f(x)dx = \int_0^b x^k \frac{2\nu\beta}{b}(1-\frac{x}{b})\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}\left[\frac{x}{b}(2-\frac{x}{b})\right]^{\nu(i+1)-1}dx$$
$$= \frac{2\nu\beta}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\beta-1}{i}\binom{\nu(1+i)-1}{j}(-1)^{i+j}2^{\nu(1+i)-j-1}b^{k+1}\left(\frac{1}{(\nu(1+i)+j+k)(\nu(1+i)+j+k+1)}\right).$$

In addition, using the expansion $e^{tx} = \sum_{m=0}^{\infty}\frac{(tx)^m}{m!}$, the moment generating function is obtained as

$$E(e^{tX}) = \int_0^b e^{tx} f(x)dx = \int_0^b e^{tx}\frac{2\nu\beta}{b}(1-\frac{x}{b})\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}\left[\frac{x}{b}(2-\frac{x}{b})\right]^{\nu(i+1)-1}dx$$
$$= \frac{2\nu\beta}{b}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\beta-1}{i}\binom{\nu(1+i)-1}{j}(-1)^{i+j}2^{\nu(1+i)-j-1}e^{t}\sum_{m=0}^{\infty}b^{m+1}\left(\frac{1}{(\nu(1+i)+j+m)(\nu(1+i)+j+m+1)}\right).$$

The variance, skewness and kurtosis measures can be obtained from the following relations:

$$Var(X) = E(X^2) - [E(X)]^2,$$
$$Skewness(X) = \frac{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3}{[Var(X)]^{3/2}},$$
$$Kurtosis(X) = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)[E(X)]^2 - 3[E(X)]^4}{[Var(X)]^2}.$$

2.6. Bonferroni and Lorenz Curves

The Bonferroni curve of the ETL distribution is given by

$$B(F(x)) = \frac{1}{\mu F(x)}\int_0^x tf(t)dt,$$

Where $\mu = E(X)$. It is clearly seen that $\int_0^x tf(t)dt = \mu - \int_x^b tf(t)dt$. from 5 and after some algebra, we have

$$\int_x^b tf(t)dt = \int_x^b t\frac{2\nu\beta}{b}(1-\frac{t}{b})\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\beta-1}{i}\binom{\nu(i+1)-1}{j}(-1)^{i+j}2^{\nu(i+1)-j-1}\left(\frac{t}{b}\right)^{\nu(i+1)+j-1}dt$$
$$= \frac{2\nu\beta}{b})\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\beta-1}{i}\binom{\nu(i+1)-1}{j}(-1)^{i+j}2^{\nu(i+1)-j-1}\int_x^b\left[\frac{t^{\nu(i+1)+j}}{b^{\nu(i+1)+j-1}} - \frac{t^{\nu(i+1)+j+1}}{b^{\nu(i+1)+j}}\right]dt$$
$$= \frac{2\nu\beta}{b})\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\beta-1}{i}\binom{\nu(i+1)-1}{j}(-1)^{i+j}2^{\nu(i+1)-j-1}$$
$$\times\left[b^2\left(\frac{1}{\nu(i+1)+j+1} - \frac{1}{\nu(i+1)+j+2}\right) - \left(\frac{x^{\nu(i+1)+j+1}}{(\nu(i+1)+j+1)b^{\nu(i+1)+j-1}} - \frac{x^{\nu(i+1)+j+2}}{(\nu(i+1)+j+2)b^{\nu(i+1)+j}}\right)\right].$$

Also, the Lorenz curve of $F(x)$ can be obtained using the relation

$$L(F(x)) = F(x)B(F(x)).$$

2.7. Order Statistics

Suppose that is a random sample of size n coming from an ETL distribution with pdf given in 3. Suppose further $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ are the corresponding order statistics. Then, the cdf and pdf of mth $(1 \le m \le n)$ order statistic, say $Y_m = X_{m:n}$, are, respectively, given by [5].

$$
\begin{aligned}
F_{X_{m:n}}(x) &= \sum_{j=m}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} = \sum_{j=m}^{n} \binom{n}{j} \left[1 - \left(1 - (\frac{x}{b})^{\nu} (2-\frac{x}{b})^{\nu} \right)^{\beta} \right]^j \left(1 - (\frac{x}{b})^{\nu} (2-\frac{x}{b})^{\nu} \right)^{\beta(n-j)} \\
&= \sum_{j=m}^{n} \sum_{i=0}^{j} \binom{n}{j} \binom{j}{i} (-1)^i \left[1 - (\frac{x}{b})^{\nu} (2-\frac{x}{b})^{\nu} \right]^{\beta(n+i-j)} \\
&= \sum_{j=m}^{n} \sum_{i=0}^{j} \sum_{k=0}^{\infty} \binom{n}{j} \binom{j}{i} \binom{\beta(n+i-j)}{k} (-1)^{i+k} \left[\frac{x}{b} (2-\frac{x}{b}) \right]^{k\nu},
\end{aligned}
$$

And

$$
f_{X_{m:n}}(x) = \frac{1}{Beta\ (m, n-m+1)} [F(x)]^{m-1} [\overline{F}(x)]^{n-m} f(x)
$$

$$
= \frac{2\nu\beta}{bBeta\ (m, n-m+1)} (1-\frac{x}{b}) \left(\frac{x}{b}(2-\frac{x}{b}) \right)^{\nu-1} \left(1 - (\frac{x}{b})^{\nu} (2-\frac{x}{b})^{\nu} \right)^{\beta(n-m+1)-1} \left(1 - \left(1 - (\frac{x}{b})^{\nu} (2-\frac{x}{b})^{\nu} \right)^{\beta} \right)^{m-1},
$$

Where $Beta\ (a, b)$ is the complete beta function.

2.7. Maximum Likelihood Estimation of the Parameters

Here, we discuss maximum likelihood estimation of the parameters of the ETL distribution. Suppose $x_1, x_2, \ldots, x_n$ is a random sample of size n from the ETL distribution. The log-likelihood function is

$$
\begin{aligned}
\ell = \ln(L) = n \ln\left(\frac{2\nu\beta}{b} \right) + \sum_{i=1}^{n} \ln(1-\frac{x_i}{b}) + (\nu-1) \sum_{i=1}^{n} \ln \frac{x_i}{b} + (\nu-1) \sum_{i=1}^{n} \ln(2-\frac{x_i}{b}) \\
+ (\beta-1) \sum_{i=1}^{n} \ln\left(1 - (\frac{x_i}{b})^{\nu} (2-\frac{x_i}{b})^{\nu} \right).
\end{aligned}
$$

The MLEs of ν, b and β say $\hat{\nu}$, $\hat{b}$ and $\hat{\beta}$, respectively, can be obtained as the solutions of the non-linear equations

$$
\frac{\partial \ell}{\partial \nu} = \frac{n}{\nu} + \sum_{i=1}^{n} \ln \frac{x_i}{b} + \sum_{i=1}^{n} \ln(2-\frac{x_i}{b}) - (\beta-1) \sum_{i=1}^{n} \frac{\left(\frac{x_i}{b}(2-\frac{x_i}{b}) \right)^{\nu} \left[\ln \frac{x_i}{b} + \ln(2-\frac{x_i}{b}) \right]}{1 - (\frac{x_i}{b})^{\nu} (2-\frac{x_i}{b})^{\nu}} = 0,
$$

$$
\frac{\partial \ell}{\partial b} = \frac{-n}{b} + \sum_{i=1}^{n} \frac{x_i}{b^2} \left(\frac{1}{1-x_i/b} + \frac{2(\nu-1)(x_i-b)}{x_i(2-x_i/b)} \right) - (\beta-1) \sum_{i=1}^{n} \frac{\frac{\nu}{b} \left[(\frac{x_i}{b})^{\nu+1} (2-\frac{x_i}{b})^{\nu-1} - \left(\frac{x_i}{b}(2-\frac{x_i}{b}) \right)^{\nu} \right]}{1 - (\frac{x_i}{b})^{\nu} (2-\frac{x_i}{b})^{\nu}} = 0,
$$

$$
\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left(1 - (\frac{x_i}{b})^{\nu} (2-\frac{x_i}{b})^{\nu} \right) = 0.
$$

Numerical methods such Newton-Raphson one can be applied to solve the above equations.

3. Application

Here, we illustrate the power of the ETL distribution by using a real data set taken from Cordeiro and Brito [6]. The data set includes total milk production in the first birth of 107 cows from SINDI race. Actually the original data is not in the interval (0, 1), therefore Cordeiro and Brito [6] made a transformation given by $x_i = [y_i - \min(y_i)]/[\max(y_i) - \min(y_i)]$, for i=1,…, 107. These data (refered as the milk production data) are available in Table 11 of Cordeiro and Brito [6].

We fitted the ETL distribution as well as the Topp-Leone to the milk production data. Table 1 lists the parameter estimates, their standard errors, the negative log-likelihood values, the values of the Akaike information criterion (AIC), the Kolmogorov-Smirnov (K-S) test statistic and the corresponding p-values. Note that since the data are transformed to (0, 1), the parameter b is known and equals one. From Table 1, we can conclude that the ETL distribution provides the better fit than the Topp-Leone distribution. The histogram and probability plots shown in Figures 3 and 4 confirm this observation. The fitted pdf of the ETL distribution captures the empirical histogram better than the pdf of the Topp-Leone distribution. The plotted points for the ETL distribution are closer to the diagonal line in the probability plot.

Table 1. The MLE's and the goodness of fit statistics for the milk production data

Model	MLE	Error	-LL	AIC	K-S	*p*-value
ETL	$\hat{\nu} = 2.4683$ $\hat{\beta} = 1.3050$	0.3016 0.1792	-23.3428	-42.5702	0.0950	0.2886
Topp-Leone	$\hat{\nu} = 2.0802$	0.2011	-21.5262	-41.0524	0.0972	0.2638

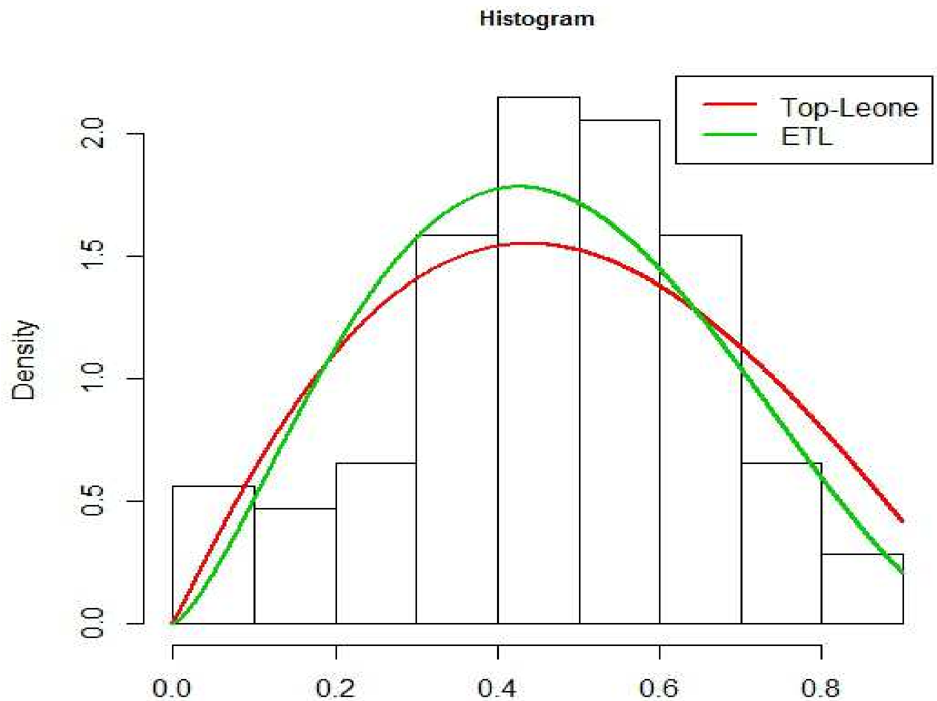

Fig 3. Histogram of the milk production data and the fitted pdfs of the ETL and the Topp-Leone models

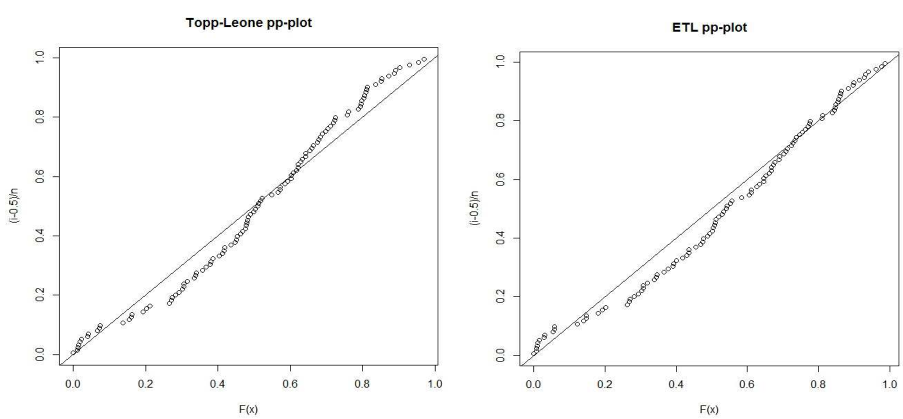

Fig. 4. Probability plots for the fits of the ETL and the Topp-Leone distributions

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