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The Kanwal and Liu method for the solution of non linear volterra integral equations

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ABSTRACT

In this study, Kanwal and Liu method for the solution of Fredholm integral equation is used to solve nonlinear Volterra integral equations apply examples that illustrate the pertinent features of the method are presented. A considerable advantage of the method is that the solution is expressed as a truncated Taylor series. Furthermore, after calculation of the series coefficients, the solution y(x) can be easily evaluated for arbitrary values of x at low computation effort. To get the best approximating solution of the equation, we take more from the Taylor expansion of function; that is, the truncation limit N must be chosen large enough. As a result, the computational complexity is reduced. An interesting feature of the method is that we get analytical solution in many cases.
KEYWORDS: Kanwal and Liu Method, Nonlinear Volterra–Fredholm.

1. INTRODUCTION

In this study, Kanwal and Liu method is applied for the solution of Fredholm integral equation

$$y(x) = f(x) + \lambda \int^{x} k(x,t)[y(t)]^{p}\,dt. \quad (1)$$

where p is positive integer, $k(x,t)$, $f(x)$,are functions having nth derivatives and λ, a are constants. Kanwal and Liu method, algebraic technique to solve the integral equations.

This method, first by Kanwal and Liu was used for solving linear Fredholm integral equation [1]. The technique is based on, first, differentiating both sides of the integral equation n times and then substituting the Taylor series for the unknown function in the resulting equation and later, transforming to a matrix equation [3].

Numerical Solution of Integral Equation by Kanwal and Liu METHOD is expressed in the form

$$y(x) = \sum_{n=0}^{N} \frac{1}{n!}\, y^{(n)}(c)(x-c)^{n}, \quad (2)$$

which is a Taylor polynomial of degree N at $x = c$, where $y^{(n)}(c), n = 0,1,\dots,N$, are coefficients of be determined.

2. Method of solution

To obtain the solution of equation (1) in the form of expression (2) we first differentiate it n times with respect to x:

$$y^{(n)}(x) = f^{(n)}(x) + \lambda \frac{d^{n}}{dx^{n}} \int_{a}^{x} k(x,t)\,[y(t)]^{p}\,dt. \quad (3)$$

<u>At the end of first page you should mention:</u>

Substituting the expression $Y(t) = [y(t)]^{p}$ in Eq. (4), we obtain:

$$y^{(n)}(x) = f^{(n)}(x) + \lambda \frac{d^{n}}{dx^{n}} \int_{a}^{x} k(x,t)\,Y(t)\,dt. \quad (4)$$

For $n = 0$

$$y^{(0)}(x) = f^{(0)}(x) + \lambda \int_{a}^{x} k(x,t)\,Y(t)\,dt.$$

By applying successively n times the Leibnitz's rule (dealing with differentiation of integrals) to the integral(4), we have, for $n \geq 1$

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$$y^{(n)}(x) = f^{(n)}(x) + \lambda \sum_{i=0}^{n-1} [h_i(x)Y(x)]^{(n-i-1)} + \lambda \int_a^x \frac{\partial^{(n)} k(x,t)}{\partial x^n} Y(t)dt \qquad (5)$$

where

$$h_i(x) = \frac{\partial^{(i)} k(x,t)}{\partial x^i} \quad t = x \ . \qquad (6)$$

From the Leibnitz's rule (dealing with differentiation of products of functions), we evaluate $[h_i(x)Y(x)]^{(n-i-1)}$ and substitute it in Eq(5). Thus we have:

$$y^{(n)}(x) = f^{(n)}(x) + \lambda \sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(x) Y^{(m)}(x) + \lambda \int_a^x \frac{\partial^{(n)} k(x,t)}{\partial x^n} Y(t)dt \qquad (7)$$

(7)

Note that in Eq (7) [2].

$$\sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} (\cdots) = \sum_{i=0}^{n-1} \sum_{m=0}^{n-i-1} (\cdots).$$

Substituting the Taylor expansion $Y(t)$ at $x = c$, i.e.

$$Y(t) = \sum_{m=0}^{N} \frac{1}{m!} Y^{(m)}(c)(t-c)^m$$

in the Eq (7). The result is

$$y^{(n)}(c) = f^{(n)}(c) + \lambda \sum_{m=0}^{n-1} \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(c) Y^{(m)}(c) + \lambda \int_a^c \frac{\partial^{(n)} k(x,t)}{\partial x^n} \left[\sum_{m=0}^{N} \frac{1}{m!} Y^{(m)}(c)(t-c)^m \right] dt$$

or briefly

$$y^{(n)}(c) - f^{(n)}(c) + \lambda \left\{ \sum_{m=0}^{n-1} (H_{nm} + T_{nm}) Y^{(m)}(c) + \sum_{m=n}^{N} T_{nm} Y^{(m)}(c) \right\} \qquad (8)$$

where for $n = 0$

$$\sum_{m=0}^{n-1}{}' (H_{nm} + T_{nm}) Y^{(m)}(c) = 0$$

for $n = 1,2,\ldots; m = 0,1,\ldots,n-1 (n > m)$

$$H_{nm} = \sum_{i=0}^{n-m-1} \binom{n-i-1}{m} h_i^{(n-m-i-1)}(c), \qquad (9)$$

for $n \le m$, $H_{nm} = 0.$

$$T_{nm} = \frac{1}{m!} \int_a^c \frac{\partial^{(n)} k(x,t)}{\partial x^n} (t-c)^m dt \qquad (10)$$

The quantities $Y^{(m)}(c)(m = 0,1,2,\cdots,N)$ in Eq. (10) can be found from the permutation relation

$$Y^{(m)}(c) = \sum_{t_1+t_2+\cdots+t_p=m} \binom{m}{t_1 t_2 \cdots t_p} y^{(t_1)}(c) y^{(t_2)}(c) \cdots y^{(t_p)}(c), \qquad (11)$$

The system (8) can be put in a matrix form as

$$-\begin{bmatrix} \lambda T_{00} & \lambda T_{01} & \lambda T_{02} & \cdots & \lambda T_{0N} \\ \lambda (H_{10}+T_{10}) & \lambda T_{11} & \lambda_1 T_{12} & \cdots & \lambda_1 T_{1N} \\ \lambda_1 (H_{20}+T_{20}) & \lambda_1 (T_{21}+T_{21}) & \lambda_1 T_{22} & \cdots & \lambda_1 K_{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1 (H_{N0}+T_{N0}) & \lambda_1 (H_{N1}+T_{N1}) & \lambda_1 (H_{N2}+T_{N2}) & \cdots & \lambda_1 T_{NN} \end{bmatrix} \tag{12}$$

$$\times \begin{bmatrix} Y^{(0)}(c) \\ Y^{1}(c) \\ Y^{(2)}(c) \\ \vdots \\ Y^{(N)}(c) \end{bmatrix} = \begin{bmatrix} f^{(0)}(c) \\ f^{(1)}(c) \\ f^{(2)}(c) \\ \vdots \\ f^{(N)}(c) \end{bmatrix}.$$

which is a algebraic system of $N+1$ nonlinear equation for $N+1$ unknowns $y^{(0)}(c), y^{(1)}(c), \ldots, y^{(N)}(c)$. These can be solved numerically by standard methods. From this nonlinear system, the unknown Taylor coefficients $y^{(n)}(c)(n=0,1,\ldots,N)$ are determined and substituted in (2).

3. Numerical Examples

Example 1.

Let us first consider the nonlinear Volterra integral equation

$$y(x) = -\frac{1}{2}x^6 + x^2 + \int_0^x (x-t)\,[y(t)]^2 dt$$

and approximate the solution $y(x)$ by the Taylor polynomial

$$y(x) = \sum_{n=0}^{5} \frac{1}{n!} y^{(n)}(0) x^n$$

$a = 0, c = 0, \lambda = 1, N = 5.$

$f(x) = -\frac{1}{2}x^6 + x^2, k(x,t) = x - t.$

First, we find the coefficients H_{nm} from (6) and (9), the coefficients T_{nm} from (12).thus

$H_{10} = 0$

$H_{20} = 1 \quad H_{21} = 0$

$H_{30} = 0 \quad H_{31} = 1 \quad H_{32} = 0$

$H_{40} = 0 \quad H_{41} = 0 \quad H_{42} = 1 \quad H_{43} = 0$

$H_{50} = 0 \quad H_{51} = 0 \quad H_{52} = 0 \quad H_{53} = 1 \quad H_{54} = 0.$

Since $a = c$, for $n, m = 0,1,2,3,4,5.$

$T_{nm} = 0$

then we get the derivation values of the function $f(x)$ at $x = 0$ as

$f^{(0)}(0) - 0, f^{(1)}(0) - 0, f^{(2)}(0) - 2, f^{(3)}(0) - 0, f^{(4)}(0) - 0, f^{(5)}(0) - 0.$

Then, for N = 5, the matrix equation (12)

$$\begin{bmatrix} y^{(0)}(0) \\ y^{(1)}(0) \\ y^{(2)}(0) \\ y^{(3)}(0) \\ y^{(4)}(0) \\ y^{(5)}(0) \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y^{(0)}(0) \\ Y^{(1)}(0) \\ Y^{(2)}(0) \\ Y^{(3)}(0) \\ Y^{(4)}(0) \\ Y^{(5)}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{13}$$

From the obtained equation system (13) and relation (11), the coefficients

$y^{(n)}(0)\ (n = 0,1,\ldots,5)$

are found as

$y^{(0)}(0) = 0, y^{(1)}(0) = 0, y^{(2)}(0) = 2, y^{(3)}(0) = 0, y^{(4)}(0) = 0, y^{(5)}(0) = 0.$

Substituting these coefficients in (2) we have the solution $y(x) = x^2$ which is an exact solution.

Example 2. consider the nonlinear Volterra integral equation

$$y(x) = e^{x} - \frac{1}{3}e^{3x} + \frac{1}{3} + \int_{0}^{x} [y(t)]^3 \, dt \qquad (14)$$

which has the exact solution $y(x) = e^{x}$.

So that

$a = 0, c = 0, \lambda = 1.$

$f(x) = e^{x} - \frac{1}{3}e^{3x} + \frac{1}{3}, K(x,t) = 1.$

we find solution of integral equation (14) for $N = 5,6,8,9$.

Since $a = c$, Then equation (8) becomes as follows:

$$y^{(0)}(0) = f^{(0)}(0), \qquad (15)$$

Now, we find the coefficients H_{nm} from (6) and (9) for $N = 5$, so

$H_{10} = 1$

$H_{20} = 0 \quad H_{21} = 1$

$H_{30} = 0 \quad H_{31} = 0 \quad H_{32} = 1$

$H_{40} = 0 \quad H_{41} = 0 \quad H_{42} = 0 \quad H_{43} = 1$

$H_{50} = 0 \quad H_{51} = 0 \quad H_{52} = 0 \quad H_{53} = 0$

also

$$y^{(n)}(0) = f^{(n)}(0) + \lambda \sum_{m=0}^{n-1} H_{nm} Y^{(m)}(0), \qquad n = 1,2,\dots,N.$$

$f^{(0)}(0) = 1, f^{(1)}(0) = 0, f^{(2)}(0) = -2, f^{(3)}(0) = -8, f^{(4)}(0) = -26, f^{(5)}(0) = -80.$

We obtained the coefficients by Equation (15) as follows:

$y^{(0)}(0) = f^{(0)}(0).$

The result is

$y^{(0)}(0) = 1.$

In addition, The coefficient obtained by applying (11),(15) AS:

$y^{(1)}(0) = f^{(1)}(0) + H_{10}Y^{(0)}(0).$

The result is

$y^{(1)}(0) = 1.$

Similarly, For n=2,3,4,5 :

$y^{(2)}(0) = f^{(2)}(0) + H_{20}Y^{(0)}(0) + H_{21}Y^{(1)}(0)$

$y^{(2)}(0) = -2 + 0 + 3 = 1.$

$y^{(3)}(0) = f^{(3)}(0) + H_{30}Y^{(0)}(0) + H_{31}Y^{(1)}(0) + H_{32}Y^{(2)}(0),$

$y^{(3)}(0) = -8 + 0 + 0 + 9 = 1.$

$y^{(4)}(0) = f^{(4)}(0) + H_{40}Y^{(0)}(0) + H_{41}Y^{(1)}(0) + H_{42}Y^{(2)}(0) + H_{43}Y^{(3)}(0),$

$y^{(4)}(0) = -26 + 0 + 0 + 0 + 27 = 1.$

$y^{(5)}(0) = f^{(5)}(0) + H_{50}Y^{(0)}(0) + H_{51}Y^{(1)}(0) + H_{52}Y^{(2)}(0) + H_{53}Y^{(3)}(0) + H_{54}Y^{(4)}(0),$

$y^{(5)}(0) = -80 + 0 + 0 + 0 + 0 + 81 = 1.$

Substituting these coefficients in (18) we have the solution

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$

which is the Taylor expansions of e^{x}.

Similarly, We obtained the coefficients for N=5,8,9 . thus

$y^{(0)}(0) = 1, y^{(1)}(0) = 1, y^{(2)}(0) = 1, y^{(3)}(0) = 1, y^{(4)}(0) = 1, y^{(5)}(0) = 1.$

and

$y^{(6)}(0) = f^{(6)}(0) + H_{60}Y^{(0)}(0) + H_{61}Y^{(1)}(0) + H_{62}Y^{(2)}(0) + H_{63}Y^{(3)}(0) + H_{64}Y^{(4)}(0)$
$+H_{65}Y^{(5)}(0)$

The result is

$y^{(6)}(0) = -242 + 0 + 0 + 0 + 0 + 0 + 243 = 1.$

then

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}.$$

allso

$$y^{(8)}(\circ) = f^{(8)}(\circ) + H_{8\circ}Y^{(\circ)}(\circ) + H_{81}Y^{(1)}(\circ) + H_{82}Y^{(2)}(\circ) + H_{83}Y^{(3)}(\circ) + H_{84}Y^{(4)}(\circ)$$

$$+H_{85}Y^{(5)}(\circ) + H_{86}Y^{(6)}(\circ) + H_{87}Y^{(7)}(\circ).$$

where

$$y^{(8)}(\circ) = -2186 + \circ + \circ + \circ + \circ + \circ + \circ + \circ + 2187 = 1$$

then

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}.$$

Eventually

$$y^{(9)}(\circ) = f^{(9)}(\circ) + H_{9\circ}Y^{(\circ)}(\circ) + H_{91}Y^{(1)}(\circ) + H_{92}Y^{(2)}(\circ) + H_{93}Y^{(3)}(\circ) + H_{94}Y^{(4)}(\circ)$$

$$+H_{95}Y^{(5)}(\circ) + H_{96}Y^{(6)}(\circ) + H_{97}Y^{(7)}(\circ) + H_{98}Y^{(8)}(\circ).$$

The result is

$$y^{(9)}(\circ) = -656\circ + \circ + \circ + \circ + \circ + \circ + \circ + \circ + \circ + 6561 = 1$$

then

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}.$$

Compare the solutions obtained with the exact solution is given in Table 1.

Table 1. result of example 1

r	x_r	Exact $y(x) = e^x$	Approximation for N $=5$	Approximation for N $=6$	Approximation for N $=8$	Approximation for N $=9$
∘	∘	1	1	1	1	1
1	∘.2	1.2214∘	1.2214∘	1.2214∘	1.2214∘	1.2214∘
2	∘.4	1.49182	1.49181	1.49182	1.49182	1.49182
3	∘.6	1.82211	1.822∘4	1.82211	1.82211	1.82211
4	∘.8	2.22554	2.22513	2.22549	2∘22554	2.22554
5	1	2.71828	2.71666	2.7∘8∘5	2.71827	2.71828

As you can see, with increasing N solution of integral equation is more accurate.

Conclusions

Nonlinear integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed. In the examples we saw, Kanwal and Liu method is efficient for solving nonlinear integral equations. A considerable advantage of the method is that the solution is expressed as a truncated Taylor series. Furthermore, after calculation of the series coefficients, the solution y(x) can be easily evaluated for arbitrary values of x at low computation effort.

To get the best approximating solution of the equation, we take more from the Taylor expansion of function; that is, the truncation limit N must be chosen large enough. For computational efficiency, some estimate for N, the degree of the approximating polynomial (the truncation limit of Taylor series) to $y(x)$, should be available. Because the choice of N determines the precision of the solution $y(x)$. if N is chosen too large, unnecessary 1abour may be done; but if N is taken a small value, the solution will not be sufficiently accurate. Therefore N must be chosen sufficiently large to get a reasonable approximation.

If $a = c$, then $T_{nm} = \circ$(n,m=0,1,2,…,N). As a result, the computational complexity is reduced. An interesting feature of the method is that we get analytical solution in many cases, as demonstrated in Example (1).

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