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Exact solution of Fredholmintegro-differential equations using optimal homotopy asymptotic method

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ABSTRACT

The aim of this paper is to present the analytical solutions of nonlinear Fredholmintegro-differential equations of second kind with different orders by using powerful analytical technique namely optimal homotopy asymptotic method (OHAM). Here, we consider the linear, non-linear Fredholmintegro-differential equations of second kind and compared the results with Homotopy perturbation method (HPM), Homotopy analysis method (HAM), CAS wavelet method, Differential transformation (DT) method, Legendre polynomial.These results reveals that the OHAM is very effective and simple and in these examples leads to the exact solutions.

KEYWORDS: Integro-differential equations, Optimalhomotopy asymptotic method, series solution.

1. INTRODUCTION

Scientific investigation and modeling of physical problems expressed in terms of mathematical equations such as differential equations, integral and integro-differential equations. Integro-differential equations have numerous applications in adverse areas of science and engineering. It rises unsurprisingly in number of models varying from applied mathematics, biological science, physical and other disciplines, such as torsion of wire, physiology, theory of elasticity, fluid dynamics, epidemiology, population problems, Bernoulli problem, oscillating magnetic field and ecological sciences etc.

Most of the problems of real world are either described by linear or by non-linear differential and integro-differential equations. The solutions of nonlinear physical phenomenon are still very difficult in comparison with linear problems. In particular, getting an exact analytical solution of a given non linear problem is often more hard as compared to getting a numerical solution. There are number of ways in which author addressed the method to solveintegro differential equations such as Adomian`s decomposition [1], HPM [2], HAM [3], CAS wavelet method [4], DT method [5], Bernstein series solutions [6], Legendre polynomial [7], variational Iteration Method [8], numerical approximation technique that uses Schauder bases[9]. These methods has its intrinsic benefits and drawbacks, so the researchers are always inhunt for another more general, easier and more accurate method.

Motivated by the literature review, the authors are interested to obtain semi-analytic solutions of Fredholmintegro-differential equations of second kind by using OHAM, which is newlypresented by Marinca and Herisanu[10]. The advantage of OHAM is in the convergence criteria, which is more elastic to control. In series of papers, authors [11-18] have applied this method successfully to obtain the solutions of currently important problems in science and also shown its effectiveness, generalization and reliability. The exact solutions of integro-differential equations showed that OHAM is beneficial for integro-differential equations also, which shows its potential, reliability andvalidity in modern science.

The paper is distributed in 4 sections. Section 2 comprises theformulation of OHAM for Fredholmintegro differential equation, the application of OHAM to integro-differential equations with examples is expressed in Section 3. Concluding remarks are in Section 4.

2. Application of OHAM to Fredholmintegro differential equation

OHAM was introduced by Marinca and Herisanu [10,11]. Here we are going to propose it extension to Fredholmintegro differential equation. The general form of Fredholmintegro-differential equation is

$$\sum_{j=1}^{n} a_j(x)D^{(j)}u(x) + \int_m^n k(x,t)F(x,t,u(t))dt + f(x) = 0 \quad (2.1)$$

With boundary conditions $B\left(u(x), \frac{d^{j-1}u(x)}{d^{j-1}x}\right) = 0$, here j-represents the jth derivative w.r.t. x. Here $a_j(x), f(x), F(x,t,u(t))$ are known and continuous function. The homotopy of the given problem is

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$$(1-q)\left(\sum_{j=1}^{n} a_j(x)D^{(j)}u(x)+f(x)\right)=H(q,c_l)\left(\sum_{j=1}^{n} a_j(x)D^{(j)}u(x)+\int_m^n k(x,t)F(x,t,u(t))dt+f(x)\right) \quad (2.2)$$

where $q \in [0,1]$ is an embedding parameter and $H(q,c_i)$ is a series of auxiliary parameters, $H(0)=0$ for $q \neq 0$, the series can be written as

$$H(q,c_i)=\sum H_j(q,c_i)$$

Here the parameters c_i help to control the convergence of the solution. Using the Taylor series expansion, we have

$$u(x;q,c_i)=u_0(x)+\sum_{k=1}^{\infty} u_k(x;c_i)q^k, \qquad i=1,2,3,\ldots \quad (2.3)$$

To demonstrate the method,we suppose $F(x,t,u(t))=u(t)$. Substituting (3) in homotopy equation, we have the solution set

$$O(q^0):\ \sum_{j=1}^{n} a_j(x)D^{(j)}u_0(x)=-f(x) \quad (2.4)$$

$$O(q^1):\ \sum_{j=1}^{n} a_j(x)D^{(j)}u_1(x)=(1+c_1)\sum_{j=1}^{n} a_j(x)D^{(j)}u_0(x)+c_1\int_m^n k(x,t)u_0(t)dt+(1+c_1)f(x) \quad (2.5)$$

$$O(q^2):\ \sum_{j=1}^{n} a_j(x)D^{(j)}u_2(x)=\begin{array}{c}(1+c_1)\sum_{j=1}^{n} a_j(x)D^{(j)}u_1(x)+c_2\sum_{j=1}^{n} a_j(x)D^{(j)}u_0(x)\\ +c_1\int_m^n k(x,t)u_1(t)dt+c_2\int_m^n k(x,t)u_0(t)dt+c_2 f(x)\end{array} \quad (2.6)$$

Similarly other problem can be generated.
By using (2.4)-(2.6) in (2.3), we have the approximate solution in terms of the auxiliary constants. The evaluate these constant, we form residual equation as

$$R(x,c_i)=\sum_{j=1}^{n} a_j(x)D^{(j)}u(x)+\int_m^n k(x,t)u(t)dt+f(x) \quad (2.7)$$

The exact solution occurs, when $R=0$, otherwise we will optimize the approximate solution using auxiliary parameters. This can be done by calculating the values of these parameters by either of the techniques namely; Galerkin method, Least square method and Ritz method etc.

3. Numerical treatment

To demonstrate the efficiency and applicability of presented technique, we have considered four examples from various branched of sciences.

3.1 Example 1

Consider the first order Fredholmintegro differential equation [9]

$$u'(x)=1-\frac{x}{3}+\int_0^1 xtu(t)dt;\quad u(0)=0 \quad (3.1)$$

Which has the exact solution $u(x)=x$, the OHAM formulation provides the following problem set

$O(q^0)$: $u'_0(x)=1-\frac{x}{3}$, $\qquad u_0(0)=0$(3.2)

$O(q^1)$: $u_1'(x)=-c_1\int_0^1 xtu_0(t)dt$, $\qquad u_1(0)=0$(3.3)

$O(q^2)$: $u_2'(x)=(1+c_1)u_1'(x)-c_1\int_0^1 xtu_1(t)dt-c_2\int_0^1 xtu_0(t)dt$, $\qquad u_2(0)=0 \qquad$ (3.4)

By solving (3.2)-(3.4), we get the values of $u_0(x), u_1(x)$and $u_2(x)$. This will give us the analytic solution of our problem

$$u(x;c_i)=\lim_{q\to 1} u(x;c_i) \quad =\sum_{k=1}^{\infty} u_k(x;c_i), \quad (3.5)$$

By calculating, we have the solutions

$$O(q^0):\ u_0(x)=\frac{1}{6}(6x-x^2), \quad (3.6)$$

$$O(q^1):\ u_1(x)=-\frac{7c_1x^2}{48}, \quad (3.7)$$

$$O(q^2):\ u_2(x)=-\frac{7x^2}{384}(8c_1+7{c_1}^2+8c_2), \quad (3.8)$$

By substituting these solution in (3.5), we have the second order approximate solution

$$u(x;c_1,c_2)=\frac{1}{6}(6x-x^2)-\frac{7c_1x^2}{48}-\frac{7x^2}{384}(8c_1+7{c_1}^2+8c_2), \quad (3.9)$$

and using least square method, we have the values of auxiliary parameters

$$c_1 = -\frac{8}{7}, c_2 = 0 \quad (3.10)$$

Using these values, we have the exact solution of the problem. The comparison is provided in Table 1.

3.2 Example 2

Consider the Fredholmintegro differential equation [9]

$$u'(x) = xe^x + e^x - x + \int_0^1 xtu(t)dt; \quad u(0) = 0 \quad (3.11)$$

Which has the exact solution $u(x) = xe^x$, using OHAM formulation, we have the following problem set and their solutions as

$O(q^0)$: $u'_0(x) = xe^x + e^x - x, \qquad u_0(0) = 0$ (3.12)

$solution$: $u_0(x) = \frac{1}{2}(2e^x - x)x,$ (3.13)

$O(q^1)$: $u_1'(x) = -c_1 \int_0^1 xtu_0(t)dt, \qquad u_1(0) = 0$ (3.13)

$solution$: $u_1(x) = -\frac{5c_1x^2}{12},$ (3.14)

$O(q^2)$: $u_2'(x) = (1 + c_1)u_1'(x) - c_1 \int_0^1 xtu_1(t)dt - c_2 \int_0^1 xtu_0(t)dt, \qquad u_2(0) = 0$ (3.15)

$solution$: $u_2(x) = -\frac{5x^2}{72}(6c_1 + 5c_1{}^2 + 6c_2),$ (3.16)

By substituting these solution in (3.5), we have the second order approximate solution

$$u(x; c_1, c_2) = \frac{1}{2}(2e^x - x)x - \frac{5c_1x^2}{12} - \frac{5x^2}{72}(6c_1 + 5c_1{}^2 + 6c_2), \quad (3.17)$$

and using least square method, we have the values of auxiliary parameters

$$c_1 = -\frac{6}{5}, c_2 = 0$$

Using these values we have the exact solution of the problem. The comparison is provided in Table 2.

3.3 Example 3

Consider the nonlinear Fredholmintegro differential equation [3]

$$u''(x) = e^x - 2 + \int_{-1}^1 e^{-4t}u^2(t)\left(u'(t)\right)^2 dt; \quad u(0) = 0, u'(0) = 1 \quad (3.18)$$

Which has the exact solution $u(x) = e^x$, using OHAM formulation, we have the following solutions

$$u_0(x) = e^x - x^2 \quad (3.19)$$

$u_1(x) = -4.813761023c_1$, (3.20)

$$u_1(x) = -8.50377204 \times 10^{-6}x^2(566073.6195c_1 + 1.2780408278 \times 10^7c_1{}^2 + 5.660731235c_2), \quad (3.21)$$

Using (3.19)-(3.21) in (3.5), we have the second order approximate solution and the values of the parameters are

$$c_1 = 0.10029237760188, \quad c_2 = -0.6354175782296$$

Using the values of $c_1's$ in $u(x; c_1, c_2)$ we have the second order solution. The comparison with previous techniques is given in Table 3.

3.4 Example 4

Consider the third order Fredholmintegro differential equation [7]

$$\begin{gathered} u'''(x) - u'(x) = 2x(\cos(1) - \sin(1)) - 2\cos(x) + \int_{-1}^1 xtu(t)dt; \\ u(0) = 0, u'(0) = 1, u''(0) - 2u'(0) = -2 \end{gathered} \quad (3.22)$$

Which has the exact solution $u(x) = \sin(x)$, using OHAM formulation, we have the following solutions

$u_0(x) = e^{-x}\left(\left(1 + e^{2x} - e^x(2 + x^2)\right)(\cos(1) - \sin(1)) + e^x \sin(x)\right);$ (3.23)

$u_1(x) = e^{-x}\left(\left(1 + e^{2x} - e^x(2 + x^2)\right)(\cos(1) - \sin(1))\right)c_1;$ (3.24)

$u_2(x) = e^{-x}\left(\left(1 + e^{2x} - e^x(2 + x^2)\right)(\cos(1) - \sin(1))\right)(c_1 + c_1{}^2 + c_2);$ (3.25)

Using (3.23)-(3.25) in $u(x; c_i)$, we have the OHAM solution and the values of the parameters are

$$c_1 = -0.11328786765750, \quad c_2 = -0.786258405623$$

Using the values of $c_1's$ in $u(x; c_1, c_2)$ we have the second order solution which is nearly exact. The comparison with previous techniques is given in Table 4.

Table 1: Comparison of absolute errors of Example 1

x	CAS Wavelet [4]	DT Method [5]	HPM [2]	SA Method [9]	OHAM
0.1	1.3492×10^{-3}	1.0012×10^{-2}	0.2351×10^{-5}	1.0179×10^{-7}	0.0
0.2	1.1569×10^{-3}	2.7865×10^{-2}	0.9259×10^{-5}	4.8277×10^{-7}	2.7755×10^{-17}
0.3	5.6715×10^{-3}	5.0873×10^{-2}	0.2083×10^{-4}	1.0178×10^{-6}	0.0
0.4	5.9311×10^{-2}	7.5536×10^{-2}	0.3704×10^{-4}	1.6193×10^{-6}	0.0
0.5	1.3233×10^{-2}	9.7189×10^{-2}	0.5787×10^{-4}	2.3089×10^{-6}	0.0
0.6	4.3929×10^{-2}	1.0955×10^{-1}	0.8333×10^{-4}	3.0935×10^{-6}	0.0
0.7	1.4120×10^{-2}	1.0413×10^{-1}	0.1134×10^{-3}	3.9780×10^{-6}	1.1102×10^{-16}
0.8	1.3451×10^{-2}	6.9451×10^{-2}	0.1481×10^{-3}	4.9957×10^{-6}	0.0
0.9	1.3205×10^{-2}	1.0013×10^{-2}	0.1857×10^{-3}	6.1354×10^{-6}	0.0

Table 2: Comparison of absolute errors of Example 2

x	CAS Wavelet [4]	DT Method [5]	SA Method [9]	OHAM
0.1	2.1794×10^{-4}	1.6667×10^{-3}	3.7900×10^{-6}	0.0
0.2	6.3855×10^{-4}	6.0939×10^{-3}	1.5160×10^{-5}	1.11022×10^{-16}
0.3	7.9137×10^{-4}	1.3202×10^{-2}	3.4110×10^{-5}	0.0
0.4	2.1559×10^{-4}	2.2914×10^{-2}	6.0640×10^{-5}	0.0
0.5	4.9936×10^{-4}	3.5158×10^{-2}	9.4750×10^{-5}	0.0
0.6	2.2173×10^{-4}	6.6965×10^{-2}	1.3644×10^{-4}	2.22045×10^{-16}
0.7	1.0565×10^{-4}	7.1243×10^{-2}	1.8571×10^{-4}	0.0
0.8	1.4323×10^{-4}	8.6398×10^{-2}	2.4256×10^{-4}	0.0
0.9	2.0775×10^{-4}	1.0810×10^{-1}	3.0699×10^{-4}	4.44089×10^{-16}

Table 3: Comparison of absolute errors of Example 3

x	HAM [3]	OHAM
-1.0	9.26591×10^{-1}	1.39444×10^{-13}
-0.8	5.93018×10^{-1}	8.92619×10^{-14}
-0.6	3.33572×10^{-1}	5.01821×10^{-14}
-0.4	1.48254×10^{-1}	2.23155×10^{-14}
-0.2	3.70235×10^{-2}	5.55112×10^{-15}
0.0	0.0	0.0
0.2	3.70635×10^{-2}	5.55112×10^{-15}
0.4	1.48254×10^{-1}	2.22045×10^{-14}
0.6	3.33570×10^{-1}	5.01821×10^{-14}
0.8	5.93013×10^{-1}	8.92619×10^{-14}
1.0	9.26582×10^{-1}	1.39444×10^{-13}

Table 4: Comparison of absolute errors of Example 4

x	Legendre Polynomial [7]	OHAM
-1.0	4.400×10^{-9}	0.0
-0.8	4.700×10^{-9}	0.0
-0.6	1.2000×10^{-9}	0.0
-0.4	2.3000×10^{-9}	5.55112×10^{-17}
-0.2	1.0000×10^{-10}	0.0
0.0	1.9921×10^{-17}	0.0
0.2	1.0000×10^{-10}	0.0
0.4	1.0700×10^{-8}	5.55112×10^{-17}
0.6	5.0100×10^{-8}	0.0
0.8	1.3537×10^{-6}	1.11022×10^{-16}
1.0	4.6560×10^{-7}	0.0

4 Concluding Remarks

The purpose of our present attempt is to find the solutions of Fredholmintegro-differential equations of second kind by using OHAM. We compare the performance of OHAM with other methods such as CAS wavelet method, DT method, HPM, HAM, numerical approximation technique that uses appropriate Schauder bases in adequate Banach spaces of continues functions. Examples including linear, nonlinear, first, second, third order Fredholmintegro differential equations of second kind are presented. The error analysis shows that even second order solutions obtained by OHAM gives exact solutions of presented problems. The obtained OHAM results shown that this approach can solve the problem easily and effectively. So it is concluded that OHAM is reliable and efficient technique for finding the solutions of integro differential equations.

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