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Semi-analytical treatment of singularly perturbed differential equations arising in biology

M.S. Hashmi[a], S. Iqbal[b] and N. Khan[c]

[a]Department of Mathematics, COMSATS Institute of Information Technology, Sahiwal, Pakistan
[b]School of Systems and Technology, University of Management Technology, Lahore, Pakistan
[c]Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan

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ABSTRACT

Optimal homotopy asymptotic method (OHAM) is used to obtain the semi-analytical solution of singularly perturbed differential equations arising in biology. The effectiveness of OHAM is ensured by its applicability in various problems of singular perturbed differential equations. Results are found to be in a good agreement with the exact solution. The tested problems for the various effect of perturbed parameter is discussed and displayed through tables and figures.
KEYWORDS: perturbed differential equations, Optimalhomotopy asymptotic method, convection dominated problem.

1. INTRODUCTION

The class of singularly perturbed two point differential equation is

$$-\epsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{1}$$

with conditions $y(0) = a, y(L) = b,$ where $0 < \epsilon \ll 1$ and $p(x), q(x)$and $f(x)$ are known functions, they are continuous and bounded in $(0,1)$ [1].The perturbed differential equations have gain its popularity due to enormous applications in adverse areas of science and engineering, such as modelling of spindling, population biology, bifurcation, stability of travelling waves, neuronal modelling for oscillation, blood flow in collapsible arteries, chemical reactor theory and reaction diffusion problem [2,3].

Various method have been proposed and implemented to solve perturbed boundary value problems. Kadalbajoo and Patidar [4] used a numerical technique to solve the perturbed differential equation. Surla and Stojanovic [5] use tension parameter and Surla et al [6,7] used difference scheme using spline to estimate the solution. Kalbajoo and Bawa [8] used variable mesh difference method, Rashidinia et al [9] used splines method for both singular and non-singular problems. Mohanty et al [10] used a family of uniform mesh tension spline for two point singularly perturbed differential equations. Lin et al. [1] used B-spline collocation method for numerical treatment of the problem for very small perturbed parameter.

There is an interesting fact that in the thin layer problem, solution exhibits multiscale character. It behaves rapidly near the thin layer and normally away from the layer. So numerical techniques presented in literature suffer a difficulty to obtain the precise solution. Hence a semi-analytical technique can play a vital role for obtaining the accurate solution.

Following the pioneering work of Marinca et al. [11,12] Several authors applied OHAM to the various complex problems modelled as differential or integral equations arising in science and engineering [13,14]. OHAM has proved it applicability, generalization and effectiveness to obtain semi-analytical solution of various problem [15-19]. The major goal of present paper is to provide a precise solution of singularly perturbed problem. Various aspects of solution regarding perturbed parameter has been discussed and displayed through graphs and figures.

2. OHAM formulation

OHAM was introduced by Marinca et al [11,12]. Here we are presenting its modified form for the perturbed differential equations:

The general two point governing equation in perturbed form is

$$-\epsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x)$$

with conditions $y(0) = a, y(L) = b,$.

First we construct the homotopy$\psi(x;q)$: $[0,L] \times [0,1] \to R$, which satisfy the following equation

Corresponding author: M.S. Hashmi, Department of Mathematics, COMSATS Institute of Information Technology, Sahiwal, Pakistan. E-mail: sadiq.hashmi@gmail.com (M.S. Hashmi)

$$H(\psi(x;q),q) = \begin{array}{c}(1-q)\left(-\epsilon y''(x) + p(x)y'(x) - f(x)\right) \\ -H(q,c_i)\left(-\epsilon y''(x) + p(x)y'(x) + q(x)y(x) - f(x)\right)\end{array} \quad (2)$$

where$q \ \epsilon \ [0,1]$ is an embedding parameter and $H(q,c_i)$ is a series of auxiliary parameters, $H(0) = 0$ for $q \neq 0$, the series can be written as

$$H(q,c_i) = \sum q^j H_j(q,c_i)$$

Here the parameters c_i help to control the convergence of the solution. Using the Taylor series expansion, we have

$$\psi(x;q,c_i) = y_0(x) + \sum_{k=1}^{\infty} y_k(x;c_i)q^k, \qquad i = 1,2,3,\dots \quad (3)$$

The above said series converges for $q \to 1$. Using in (2), we have the approximate solution

$$\tilde{y}(x;c_i) = y_0(x) + \sum_{k=1}^{\infty} y_k(x;c_i)q^k, \qquad i = 1,2,3,\dots \quad (4)$$

Substituting in (1), we have the residual of the problem

$$R(x,c_i) = -\epsilon\tilde{y}''(x) + p(x)\tilde{y}'(x) + q(x)\tilde{y}(x) - f(x) \quad (5)$$

The exact solution occurs, when $R = 0$, otherwise, we will optimize the approximate solution using auxiliary parameters. This can be done by calculating the values of these parameters by either of the techniques namely; Galerkin method, Least square method and Ritz method etc.

3. Numerical treatment

To demonstrate the efficiency and applicability of presented technique, we have considered three cases of singularly perturbed differential equations arising in biology.

3.1 Example 1

The convection dominated problem [1] arises when $p(x) = 1, q(x) = 1$ and $f(x) = 1$ for the domain set $0 < x < 1$ with boundary conditions$a = b = 0$. The exact solution of the perturbed problem is $y(x) = ((e^{\beta} - 1)e^{\alpha x})/(e^{\alpha} - e^{\beta}) + ((1 - e^{\alpha})e^{\beta x})/(e^{\alpha} - e^{\beta}) + 1$, where $\alpha = \frac{(1+\sqrt{1+4\epsilon})}{2\epsilon}$ and $\beta = \frac{(1-\sqrt{1+4\epsilon})}{2\epsilon}$.

Table 1. Optimal values of c_i for different value of ϵ using third order OHAM solution

c_i/ϵ	$\epsilon = 0.8$	$\epsilon = 0.2$	$\epsilon = 0.01$	$\epsilon = 0.0015$
c_1	-0.94061705	0.86692395	-0.8552525	0.0015325
c_2	-0.00115418	-0.00563309	0.00520673	0.0001202
c_3	0.0002240	0.00038186	0.00054846	-0.0879556

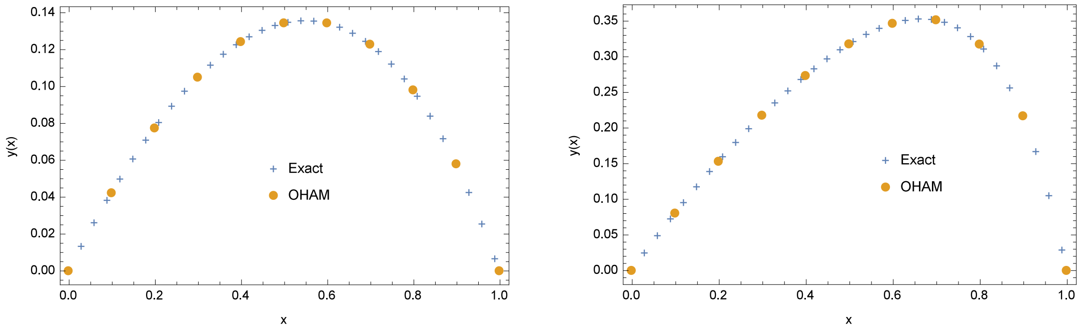
To find the solution of the problem, the value of c_i for different values of ϵ is calculated and tabulated in Table 1. The absolute pointwise error of the convection dominated problem is presented in Table 2. It is quite evident from Figure 1 & 2 that OHAM produced a reliable solution for large and small values of the perturbed parameter ϵ. The results are also compared with [1] and showed significant improvement as shown in Table 3.

Table 2:Pointwise error for differentϵ using third order OHAM solution

x/ϵ	$\epsilon = 0.8$	$\epsilon = 0.2$	$\epsilon = 0.01$	$\epsilon = 0.0015$
0.0	1.84570×10^{-15}	2.18196×10^{-17}	8.68596×10^{-15}	6.87261×10^{-15}
0.2	3.96166×10^{-9}	5.67675×10^{-8}	3.35689×10^{-6}	1.12068×10^{-5}
0.4	6.05812×10^{-9}	7.51894×10^{-8}	2.33935×10^{-6}	1.71098×10^{-4}
0.6	1.33342×10^{-9}	1.59763×10^{-7}	5.16407×10^{-7}	9.80745×10^{-5}
0.8	3.32494×10^{-9}	1.59324×10^{-8}	1.19808×10^{-6}	3.26467×10^{-4}
1.0	4.55191×10^{-15}	2.55351×10^{-15}	4.20712×10^{-15}	1.00012×10^{-15}

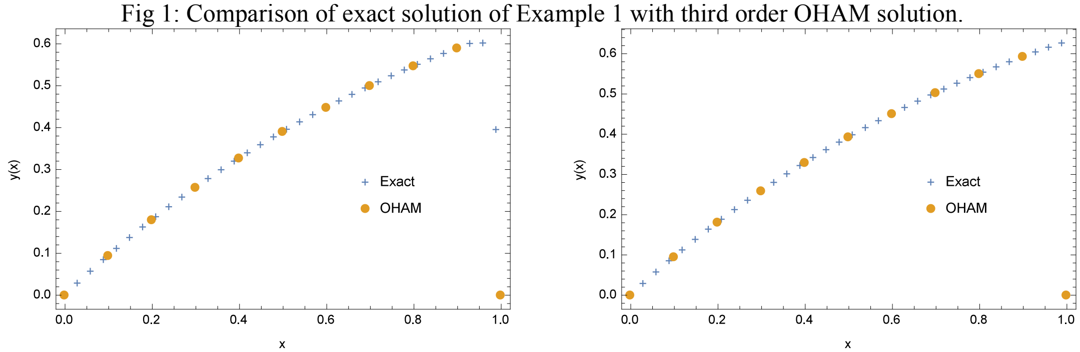
Table 3: Comparison of pointwise error of [1] with OHAM

x/ϵ	$\epsilon = 0.8$		$\epsilon = 0.2$		$\epsilon = 0.01$	
	$h = \frac{1}{128}$	3rd Order OHAM	$h = \frac{1}{128}$	3rd Order OHAM	$h = \frac{1}{128}$	3rd Order OHAM
0.0	0.0033	1.8457×10^{-15}	0.0062	2.18196×10^{-17}	0.0073	8.68596×10^{-15}
0.2	0.0029	3.96166×10^{-9}	0.0059	5.67675×10^{-8}	0.0069	3.35689×10^{-6}
0.4	0.0022	6.05812×10^{-9}	0.0061	7.51894×10^{-8}	0.0061	2.33935×10^{-6}
0.6	0.0014	1.33342×10^{-9}	0.0054	1.59763×10^{-7}	0.0054	5.16407×10^{-7}
0.8	0.0019	3.32494×10^{-9}	0.0025	1.59324×10^{-8}	0.0037	1.19808×10^{-6}
1.0	0.0034	4.55191×10^{-15}	0.0092	2.55351×10^{-15}	0.0033	4.20712×10^{-15}



(a) $\epsilon = 0.8$ (b) $\epsilon = 0.2$

Fig 1: Comparison of exact solution of Example 1 with third order OHAM solution.



(a) $\epsilon = 0.01$ (b) $\epsilon = 0.0015$

Fig 2: Comparison of exact solution of Example 1 with third order OHAM solution.

3.2 Example 2

The neuronal models of oscillation [1]is for $p(x) = p, q(x) = 1$ and $f(x) = \cos(\pi x)$ for the domain set $0 < x < 1$ with boundary conditions $a = b = 0$. The exact solution of the perturbed problem is$y(x) = \alpha\cos(\pi x) + \beta\sin(\pi x) + Ae^{\gamma x} + Be^{-\eta(1-x)}$, where $\alpha = \frac{\epsilon\pi^2+1}{p^2\pi^2+(\epsilon\pi^2+1)^2}$ and $\beta = \frac{p\pi}{p^2\pi^2+(\epsilon\pi^2+1)^2}$,$A = -\alpha\frac{(1+e^{-\eta})}{1-e^{\gamma-\eta}}$, and $B = \alpha\frac{(1+e^{-\gamma})}{1-e^{\gamma-\eta}}$. Moreover $\gamma < 0$ & $\eta > 0$, the values of α and β can found from the real roots of $-\epsilon\lambda^2 + p\lambda + 1 = 0$

Table 4. Optimal values of c_i for different value of ϵ for $p = 10^{-6}$ using 2nd order OHAM solution

c_i/ϵ	$\epsilon = 0.8$	$\epsilon = 0.2$	$\epsilon = 0.01$	$\epsilon = 0.0015$
c_1	-0.2643132	-1.00112322	-1.0000306	-1.0
c_2	-0.5412349	-1.261×10^{-6}	0.0	0.0

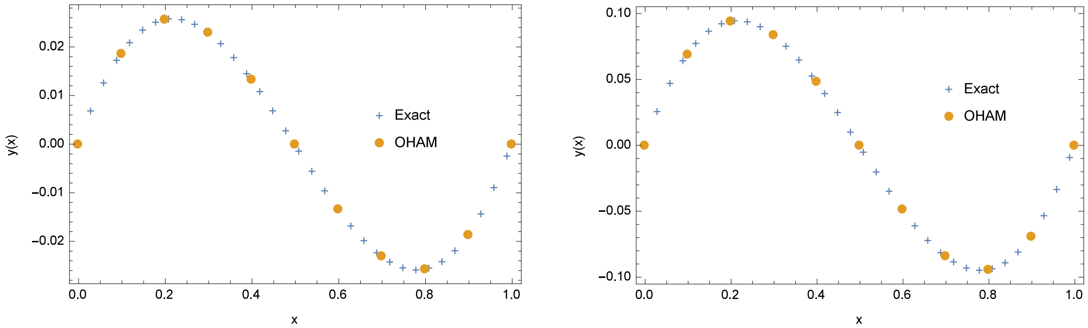
To find the solution of the problem, the value of c_i for different values of ϵ is calculated and tabulated in Table 4. The absolute pointwise error of the example 2 is presented in Table 5. It is quite evident from Figure 3 & 4 the OHAM produced a reliable solution for large and small values of the perturbed parameter ϵ. Table 6 represents the absolute error of approximate solution for $\epsilon = 0.0015$ and for different values of p.

Table 5: Pointwise error for different values of$\epsilon$and $p = 10^{-6}$using third order OHAM solution

x/ϵ	$\epsilon = 0.8$	$\epsilon = 0.2$	$\epsilon = 0.01$	$\epsilon = 0.0015$
0.0	2.77550×10^{-17}	1.11022×10^{-17}	1.11022×10^{-17}	1.11022×10^{-17}
0.2	8.15321×10^{-16}	6.93886×10^{-17}	6.66134×10^{-16}	1.11022×10^{-17}
0.4	4.66641×10^{-16}	8.32667×10^{-17}	2.27596×10^{-15}	0.0
0.6	5.08274×10^{-16}	7.63278×10^{-17}	1.66982×10^{-15}	5.55112×10^{-17}
0.8	8.39606×10^{-16}	1.38778×10^{-17}	2.22045×10^{-16}	0.0
1.0	2.77556×10^{-17}	1.19070×10^{-17}	2.76186×10^{-16}	0.0

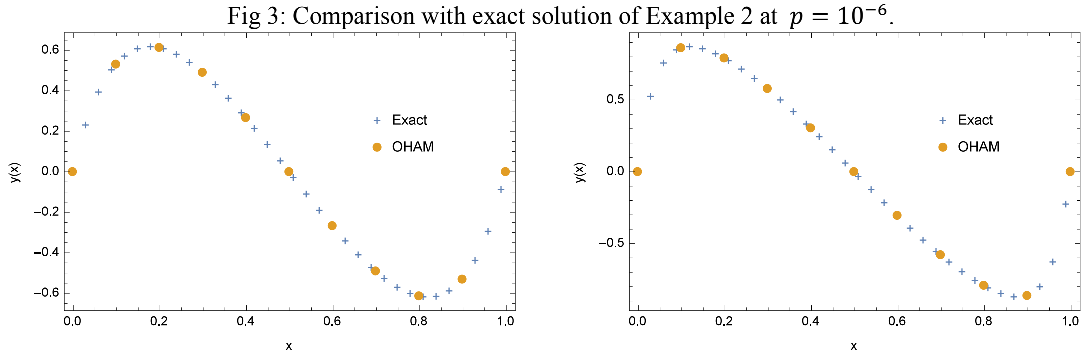
Table 6: Pointwise error for different values of p at $\epsilon = 0.0015$ second order OHAM solution

x/p	$p = 10^{-4}$	$p = 10^{-5}$	$p = 10^{-6}$
0.0	3.44169×10^{-15}	3.77476×10^{-15}	1.11022×10^{-17}
0.2	2.21141×10^{-10}	1.30342×10^{-13}	1.11022×10^{-17}
0.4	2.23121×10^{-10}	3.69704×10^{-14}	0.0
0.6	2.23127×10^{-10}	3.70259×10^{-14}	5.55112×10^{-17}
0.8	2.21407×10^{-10}	1.30342×10^{-13}	0.0
1.0	3.55227×10^{-15}	3.77475×10^{-15}	0.0



(b) $\epsilon = 0.8$ (b) $\epsilon = 0.2$

Fig 3: Comparison with exact solution of Example 2 at $p = 10^{-6}$.



(b) $\epsilon = 0.01$ (b) $\epsilon = 0.0015$

Fig 4: Comparison with exact solution of Example 2 at $p = 10^{-6}$.

3.3 Example 3

A non-homogenous perturbed equation is $p(x) = 0, q(x) = 1$ and $f(x) = -\cos^2(\pi x) - 2\epsilon\pi^2\cos(\pi x)$ with the domain set $0 < x < 1$ and boundary conditions $a = b = 0$. The exact solution of the perturbed problem is $y(x) = \frac{e^{\frac{(x-1)}{\sqrt{\epsilon}}} + e^{\frac{-x}{\sqrt{\epsilon}}}}{1-e^{\frac{-1}{\sqrt{\epsilon}}}} - \cos^2(\pi x)$ [5-9]. The approximate results are obtained by second order OHAM solution and compared with exact solution and results of various authors [5-9] and proved its authority over other methods as presented in Table 7.

Table 7: Error comparison of solutions obtained by [5-9] with 2nd order OHAM

ϵ	$N = 16$ [5]	$N = 16$ [6]	$N = 16$ [7]	$N = 16$ [8]	$N = 16$ [9]	OHAM
1/16	8.06 (-3)	4.14 (-3)	1.20 (-4)	7.09 (-3)	4.07 (-5)	7.29 (-7)
1/32	7.11 (-3)	3.68 (-3)	1.28 (-4)	5.68 (-3)	2.00 (-5)	8.49 (-6)
1/64	6.58 (-3)	3.45 (-3)	1.60 (-4)	4.07 (-3)	5.45 (-5)	2.07 (-5)
1/128	6.36 (-3)	3.43 (-3)	2.34 (-4)	6.97 (-3)	1.83 (-4)	1.78 (-4)

4. Conclusion

(a) When ϵ decreases from 0.8 to 0.0015, then error increases in examples 1.

(b) There is small error at the boundaries of the domain in example 1.

(c) The result provided by 3rd order OHAM is much better than results by [1] at $h = \frac{1}{128}$, as presented in Table 3.

(d) It is seen that pointwise error of example 2, when ϵ decreases from 0.8 to 0.0015, is almost steady at $p = 10^{-6}$.

(e) It is also worth mentioning that error decreases as p varies from 10^{-4} to 10^{-6} at the fixed value of $\epsilon = 0.0015$.

(f) In example 3, OHAM proves its reliability over other techniques [5-9].

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