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Certain Characterizations of Ordered Semigroups by Uni-Soft Generalized Bi-Ideals

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ABSTRACT

In this paper, the soft version of generalized bi-ideals in ordered semigroups is considered. The concept of uni-soft generalized bi-ideals is introduced and some related properties are investigated. Characterizations of uni-soft generalized bi-ideals are discussed. Using the notion of uni-soft generalized bi-ideals, characterizations of regular ordered semigroups (see Theorem 1, 2 and Proposition 8) completely regular and left weakly regular ordered semigroups (see Theorems 3 and 4) are provided. Moreover, a relation between uni-soft generalized bi-ideals and uni-soft bi-ideals is established.
KEYWORDS:ordered semigroups, δ-exclusive set, soft set, uni-soft generalized bi-ideal, uni-soft bi-ideal.

1. INTRODUCTION

Fuzzy set theory was initiated by Zadeh in 1965 [23]. This theory was developed in view of the fact that the classical sets are not suitable in describing real-life problems. Following the discovery of fuzzy sets, great attention has been given to generalize the basic concepts of classical algebra in a fuzzy setting, and which develops a theory of fuzzy algebra. In recent some years, great attention is shown to generalize algebraic structures of groups, rings, modules, vector spaces, etc.
Molodtsov [18] in 1999 introduced the fundamental concept of soft set which provides a natural framework for generalizing several basic notions of algebra. Many related concepts with soft sets, especially soft set operations, have also undergone tremendous studies. Later, Maji et al. [15] studied the theoretical aspect of soft set and introduced several binary operations such as intersection, union, AND-operation, and OR-operation of soft sets and investigated them in more detail. Feng et al. [4] defined the bi- intersection of two soft sets as an alternative to the definition of soft sets intersection by Maji et al. [15]. Maji at al. [16, 17] presented the theory of soft sets and fuzzy soft sets and worked on the applications of soft set theory in decision making problems. The applications of the soft theory in algebraic structures were introduce by Aktas and Cagman [19]. They presented the idea of soft groups as a generalization of the fuzzy groups. Then in 2009, Shabir at al. [20] studied soft semigroups and soft ideals over a semigroups. Ali et al. [1] introduced several operations of soft sets. Khan et al. [13, 14] characterized different classes of ordered semigroups in terms of uni-soft quasi-ideals and uni-soft ideals. The theory of soft set has also wide-ranging applications as in the following studies [25, 26].
In this paper, the notion of uni-soft generalized bi-ideals of ordered semigroups are introduced and some characterizations of uni-soft generalized bi-ideals are discussed. Using the notion of uni-soft generalized bi-ideals, we provide characterizations of regular ordered semigroups, completely regular and left weakly regular ordered semigroups. Moreover, it is shown that uni-soft generalized bi-ideal and uni-soft bi-ideal coincide in regular and left weakly regular ordered semigroups.

2. Basic definitions and preliminaries

An ordered semigoup is a semigroup $(S,\cdot)$ together with a partial orderd $\leq$ such that for all $a,b,x \in S$, $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows

$$AB = \{ab \in S \mid a \in A, b \in B\}.$$

A subset X of S is called
(1) a subsemigroup of S if $XX \subseteq X$.
(2) a right (resp., left) ideal of S if:

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(i) $(\forall a \in X)(\forall b \in S)(b \leq a \text{ implies } b \in X)$,

(ii) $XS \subseteq X$ (resp., $SX \subseteq X$).

(3) two-sided ideal (or simply ideal), if it is both a left and a right ideal of S.

(4) a generalized bi-ideal of S if:

(iii) $(\forall a \in X)(\forall b \in S)(b \leq a \text{ implies } b \in X)$,

(iv) $XSX \subseteq X$.

(5) a bi-ideal of S if it is (1) and (4).

By $R(a)$, $L(a)$, $I(a)$, and $B(a)$, we mean the principal right, left, two-sided ideal, and bi- ideal of S generated by $x(x \in S)$, respectively. Note that

$$R(a) = (a \cup aS],$$
$$L(a) = (a \cup Sa],$$
$$I(a) = (a \cup Sa \cup aS \cup SaS],$$
$$B(a) = \left(a \cup a^2 \cup aSa\right].$$

The soft set theory introduced by Molodtsov (see [18]), Cağman and Enginoğlu (see [2]) provided new definitions and various results in this theory. In what follows, let U be an common universe set and E be a set of parameters. The power set of U denoted by $P(U)$ and A, B, C are subsets of E.

The concept of a soft set is given in the following:

Definition 1.[*18*] *A soft set* (f_S, S) *over* U *is defined to be the set of ordered pairs*

$$f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\},$$

Where $f_A : E \to P(U)$ *such that* $f_A(x) = \phi$ *if* $x \notin E$. *The function* f_A *is called approximate function of the soft set* (f_S, S) *over* U.

Definition 2.[18]. Let $f_A, f_B \in S(U)$. Then

(1) f_A is called soft subset of f_B and denoted by $(f_A, S) \tilde{\subseteq} (f_B, S)$ if

$$f_A(x) \subseteq f_B(x), \forall x \in E.$$

(2)*soft union* of f_A and f_B, denoted by $(f_A \tilde{\cup} f_B, S)$ and is defined as

$$(f_A \tilde{\cup} f_B)(x) = f_{A \tilde{\cup} B}(x) = f_A(x) \cup f_B(x), \forall x \in E.$$

(3) *soft intersection* of f_A and f_B, denoted by $(f_A \tilde{\cap} f_B, S)$ and is defined as

$$(f_A \tilde{\cap} f_B)(x) = f_{A \tilde{\cap} B}(x) = f_A(x) \cap f_B(x), \forall x \in E.$$

(4)theuni-soft product, denoted by $(f_S \Diamond g_S, S)$, is defined by

$$f_S \Diamond g_S : S \to P(U), x \mapsto \begin{cases} \bigcap_{(y,z) \in A_x} \{f_S(y) \cup g_S(z)\} & \text{if } A_x \neq \varphi, \\ U & \text{if } A_x = \varphi, \end{cases}$$

where $A_x = \{(y,z) \in S \times S \mid x \leq yz\}$ for all $x, y, z \in S$.

"For a subset A of S, we define mapping χ_A as follows

$$\chi_A : S \to P(U), x \mapsto \begin{cases} U & \text{if } x \in A, \\ \varphi & \text{if } x \in S \setminus A. \end{cases}$$

Which is called the characteristic soft set over U.

For a subset A of S, the soft set (χ_A^c, S) over U in which χ_A^c is define as follows

$$\chi_A^c : S \to P(U), x \mapsto \begin{cases} \varphi & \text{if } x \in A, \\ U & \text{if } x \in S \setminus A. \end{cases}$$

For a soft set (f_S, S) over U and a subset δ of U, the δ-exclusive set of (f_S, S), denoted by $e_S(f_S; \delta)$, is defined to be the set

$$e_A(f_S;\delta)=\{x\in A|f_S(x)\subseteq\delta\}.$$

Definition 3.[5] *Let* (f_S,S) *be a soft set over* U *. Then*

(1) (f_S,S) is called a *uni-soft semigroup* over U , if

$$f_S(xy)\subseteq f_S(x)\cup f_S(y),\forall x,y\in S.$$

(2) (f_S,S) is called a *uni-soft left* $(\text{resp.},right)$ *ideal* over U if

(i) $x\le y\Rightarrow f_S(x)\subseteq f_S(y),$

(ii) $(\forall x,y\in S)(f_S(xy)\subseteq f_S(y)(\text{resp.},f_S(xy)\subseteq f_S(x))).$

If a soft set (f_S,S) over U is both a uni-soft left ideal and a uni-soft right ideal over U , we say that (f_S,S) is a uni-soft two-sided ideal over U .

Definition 4.[13]A uni-soft semigroup (f_S,S) over U is called a *uni-soft bi-ideal* over U if it satisfies

(i) $x\le y\Rightarrow f_S(x)\subseteq f_S(y),$

(ii) $(\forall x,y,z\in S)(f_S(xyz)\subseteq f_S(x)\cup f_S(z))$.

Proposition 1.[5].For subset A and B of S , the following properties hold

(i) $(\chi_A^c\Diamond\chi_B^c,S)=(\chi_{(AB]}^c,S).$

(ii) $(\chi_A^c\tilde{\cup}\chi_B^c,S)=(\chi_{A\cup B}^c,S).$

Lemma 1. Let A be non-empty subst of S , then the soft set $(\chi_{(A]}^c,S)$ over U has the following property.

$$x\le y\Rightarrow\chi_{(A]}^c(x)\subseteq\chi_{(A]}^c(y),\forall x,y\in S.$$

Proof .Let $x,y\in S$, be such that $x\le y$. If $y\notin(A]$, then $\chi_{(A]}^c(y)=U$ and so $\chi_{(A]}^c(x)\subseteq U=\chi_{(A]}^c(y)$. Let $y\in(A]$, then $\chi_{(A]}^c(y)=\phi$. Since $y\in(A]$, so there exists $z\in A$ such that $y\le z$. We have $x\le z$, hence $x\in(A]$. Then $\chi_{(A]}^c(x)=\phi$ and $\chi_{(A]}^c(x)\subseteq\varphi=\chi_{(A]}^c(y)$. .

Lemma 2. Let A be a subset of S, then $A=(A]$ if and only if the soft set (χ_A^c,S) over U has the property.

$$x\le y\Rightarrow\chi_A^c(x)\subseteq\chi_A^c(y),\forall x,y\in S.$$

Proof$\Rightarrow$. It follows from Lemma 1 .

$\Leftarrow$. Let $x\in(A]$, then there exists $y\in A$ such that $x\le y$. By hypothesis, we have $\chi_A^c(x)\subseteq\chi_A^c(y)$. Since $y\in A$, we have $\chi_A^c(y)=\phi$. Then $\chi_A^c(x)=\phi$ and $x\in A$. Hence $(A]\subseteq A$. On the other hand $A\subseteq(A]$. Therefore $A=(A]$.

Lemma 3.[5]Let A be any subset of S. Then, A is a right $(\text{left},\text{two-sided})$ ideal of S if and only if (χ_A^c,S) is an uni-soft right $(\text{left},\text{two-sided})$ over U.

Lemma 4.A soft set (f_S,S) is a uni-soft semigroup over U if and only if

$$(f_S\Diamond f_S,S)\tilde{\supseteq}(f_S,S).$$

Proof. Assume that (f_S,S) is a uni-soft semigroup over U. Let a be an element of S. If $A_a\ne\phi$, then there exists $x,y\in S$ such that $a\le xy$. Since (f_S,S) is a uni-soft semigroup over U, then we have

$$\begin{aligned}(f_S\Diamond f_S)(a)&=\bigcap_{a\le xy}\{f_S(x)\cup f_S(y)\}\\&\supseteq\bigcap_{a\le xy}f_S(xy)\\&\supseteq f_S(a).\end{aligned}$$

Thus, $(f_S\Diamond f_S,S)\tilde{\supseteq}(f_S,S)$.

Conversely, assume that $(f_S \Diamond f_S, S) \tilde{\supseteq} (f_S, S)$. . Let $x, y \in S$, we have

$$\begin{aligned} f_S(xy) &\subseteq (f_S \Diamond f_S)(xy) \\ &= \bigcap_{xy \le pq} \{f_S(p) \cup f_S(q)\} \\ &\subseteq f_S(x) \cup f_S(y). \end{aligned}$$

Hence, (f_S, S) is a uni-soft semigroupover U.

3. Uni-soft generalized bi-ideals of ordered semigroups

In this section, we define uni-soft generalized bi-ideals and study different properties of uni-soft generalized bi-ideals as a regards of uni-soft product.

Definition 5.A soft set (f_S, S) over U is called a uni-soft generalized bi-ideal over U if it satisfies.

(i) $x \le y \Rightarrow f_S(x) \subseteq f_S(y)$,

(ii) $(\forall x, y, z \in S)(f_S(xyz) \subseteq f_S(x) \cup f_S(z))$.

Example1.*Let $S = \{a, b, c, d, f\}$ be an ordered semigroup defined by the following caylays table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\le := \{(a,a),(a,b),(a,c),(a,d),(b,b),(c,c),(c,d),(d,d)\}.$$

Let (f_S, S) be a soft set over $U = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ in which f_S is given by

$$f_S : S \to P(U), x \mapsto \begin{cases} \{\bar{0}\}, & \text{if } x = a, \\ \{\bar{0}, \bar{1}\}, & \text{if } x = c, \\ \{\bar{0}, \bar{1}, \bar{2}\}, & \text{if } x \in \{b, d\} \end{cases}$$

Then by routine checking it is easy to verify that (f_S, S) is a uni-soft generalized bi-ideal over U .

Theorem 1.*A soft set (f_S, S) over U is a uni-soft generalized bi-ideal over U if and only if*

$$(f_S \Diamond \phi_S \Diamond f_S, S) \tilde{\supseteq} (f_S, S).$$

$$x \le y \Rightarrow f_S(x) \subseteq f_S(y), \forall x, y \in S.$$

Proof. Assume that (f_S, S) is a uni-soft generalized bi-ideal over U . Let x be an element of S. In the case, when $A_x \ne \phi$. Then there exists $p, q, u, v \in S$, such that $x \le pq$ and $p \le uv$. Then, since (f_S, S) is a uni-soft generalized bi-ideal over U , we have

$$f_S(x) \subseteq f_S(pq) \subseteq f_S(uvq) \subseteq f_S(u) \cup f_S(q).$$

Thus,

$$\begin{aligned} (f_S \Diamond \phi_S \Diamond f_S)(x) &= \bigcap_{x \le pq} \{(f_S \Diamond \phi_S)(p) \cup f_S(q)\} \\ &= \bigcap_{x \le pq} \left[\bigcap_{p \le uv} \{f_S(u) \cup \phi_S(v)\} \cup f_S(q) \right] \\ &= \bigcap_{x \le pq} \left[\bigcap_{p \le uv} \{f_S(u) \cup \phi\} \cup f_S(q) \right] \\ &= \bigcap_{x \le uvq} \{f_S(u) \cup f_S(q)\} \\ &\supseteq f_S(x). \end{aligned}$$

Hence, $(f_S \Diamond \phi_S \Diamond f_S, S) \tilde{\supseteq} (f_S, S)$.

Conversely, assume that $(f_S \Diamond \phi_S \Diamond f_S, S) \tilde{\supseteq} (f_S, S)$. . Let $a \in S$, then there exists $x, y, z \in S$ such that $a = xyz$. Then, we have

$$\begin{aligned} f_S(xyz) &= f_S(a) \\ &\subseteq (f_S \Diamond \phi_S \Diamond f_S)(a) \\ &= \bigcap_{a \le pq} \{(f_S \Diamond \phi_S)(p) \cup f_S(q)\} \\ &\subseteq (f_S \Diamond \phi_S)(xy) \cup f_S(z) \\ &= \bigcap_{xy \le uv} \{f_S(u) \cup \phi_S(v)\} \cup f_S(z) \\ &\subseteq f_S(x) \cup \phi_S(y) \cup f_S(z) \\ &= f_S(x) \cup \phi \cup f_S(z) \\ &= f_S(x) \cup f_S(z). \end{aligned}$$

Thus, (f_S, S) is a uni-soft generalized bi-ideal over U.

Proposition 2.A subset A of S is a generalized bi-ideal of S if and only if (χ_A^c, S) is a uni-soft generalized bi-ideal over U.

Proof.Assume that X is a generalized bi-ideal of S , that is, $XSX \subseteq X$. Then we have

$$\chi_X^c \Diamond \phi_S \Diamond \chi_X^c = \chi_X^c \Diamond \chi_S^c \Diamond \chi_X^c = \chi_{(XSX]}^c \tilde{\supseteq} \chi_X^c, \textit{since } XSX \subseteq X.$$

Furthermore, Let $x, y \in S$ be such that $S \ni x \le y \in X$. If $\chi_X^c(y) = \phi$, then $y \in X$. Since $x \le y$ and X is a generalized bi-ideal of S , we have $x \in X$. Hence $\chi_X^c(x) = \phi \subseteq \chi_X^c(y)$. If $\chi_X^c(y) = U$, then obviously, $\chi_X^c(x) \subseteq U = \chi_X^c(y)$.This means that (χ_X^c, S) is a uni-soft generalized bi-ideal over U.

Conversely, assume that (χ_X^c, S) is a uni-soft generalized bi-ideal over U. Let $x \in XSX$. Then, we have

$$\chi_X^c(x) \subseteq (\chi_X^c \Diamond \phi_S \Diamond \chi_X^c)(x) = (\chi_X^c \Diamond \chi_S^c \Diamond \chi_X^c)(x) = \chi_{(XSX]}^c(x) = \phi,$$

and so $x \in X$. Thus, $XSX \subseteq X$ and X is a generalized bi-ideal of S .

Theorem 2.A soft set (f_S, S) over U is a uni-soft generalized bi-ideal over U if and only if the non-empty δ -exclusive set of (f_S, S) is a generalized bi-ideal of S for all $\delta \subseteq U$.

Proof.First assume that (f_S, S) is a uni-soft generalized bi-ideal over U . Let $\delta \subseteq U$ be such that $e_S(f_S; \delta) \ne \phi$. Let $y \in S$ and $x, z \in e_S(f_S; \delta)$, then $f_S(x) \subseteq \delta$ and $f_S(z) \subseteq \delta$. Then by hypothesis, we have

$$f_S(xyz) \subseteq f_S(x) \cup f_S(z) \subseteq \delta.$$

It follows that $xyz \in e_S(f_S; \delta)$.

Furthermore, let $x, y \in S$ be such that $x \le y$. Let if $y \in e_S(f_S; \delta)$, then by hypothesis we have

$$f_S(x) \subseteq f_S(y) \subseteq \delta,$$

which means that $x \in e_S(f_S; \delta)$. Thus, $e_S(f_S; \delta)$ is a generalized bi-ideal of S .

Conversely, assume that the non-empty δ -exclusive set of (f_S, S) is a generalized bi-ideal of S for all $\delta \subseteq U$. Let $x, y, z \in S$ be such that $f_S(x) = \delta_x$ and $f_S(z) = \delta_z$. Taking $\delta = \delta_x \cup \delta_z$ implies that $x, z \in e_S(f_S; \delta)$. Hence $xyz \in e_S(f_S; \delta)$, and so

$$f_S(xyz) \subseteq \delta = \delta_x \cup \delta_z = f_S(x) \cup f_S(z).$$

Let $x, y \in S$, $x \leq y$ be such that $f_S(y) = \delta$, implies that $y \in e_S(f_S;\delta)$. Since $e_S(f_S;\delta)$ is a generalized bi-ideal of S for all $\delta \subseteq U$, so $x \in e_S(f_S;\delta)$. Thus

$$f_S(x) \subseteq \delta = f_S(y).$$

Therefore, (f_S, S) is a uni-soft generalized bi-ideal over U.

Proposition 3. In an ordered semigroup S, every uni-soft bi-ideal is a uni-soft generalized bi-ideal.
Proof. It is straight farward.
The converse of Proposition 3 does not hold in general as shown in the following example.

Example 2. Consider the ordered semigroup in Example 1, define a soft set (f_S, S) over $U = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ in which f_S is given by

$$f_S : S \to P(U), x \mapsto \begin{cases} \{\bar{0}\}, & \text{if } x = a, \\ \{\bar{0}, \bar{2}\}, & \text{if } x = c, \\ \{\bar{0}, \bar{2}, \bar{3}\}, & \text{if } x \in \{b, d\}. \end{cases}$$

Then (f_S, S) is a uni-soft generalized bi-ideal over U. However since

$$f_S(cc) = f_S(b) \not\subseteq f_S(c) \cup f_S(c).$$

So (f_S, S) is not a uni-soft bi-ideal over U.

Uni-soft generalized bi-ideals of regular ordered semigroups

In this section, we study different properties of uni-soft generalized bi-ideals as a regards of uni-soft product. We also characterize regular ordered semigroups in terms of uni-soft generalized bi-ideals. An element $a \in S$ is called regular, if there exists some $x \in S$ such that $a \leq axa$. An ordered semigroup S is called regular if every element of S is regular.

Proposition 4.[21].In a regular ordered semigroup S , every generalized bi-ideal is a subsemigroup, that is, a bi-ideal of S .

Proposition 5. In a regular ordered semigroup S, every uni-soft generalized bi-ideal is a uni-soft semigroup, that is, a uni-soft bi-ideal over U.

Proof. Let (f_S, S) be a uni-soft generalized bi-ideal over U . Let $a, b \in S$. Since S is regular, so there exists $x \in S$, such that $a \leq axa \Rightarrow ab \leq (axa)b$. Then

$$\begin{aligned} f_S(ab) &\subseteq f_S((axa)b) = f_S(a(xa)b) \\ &\subseteq f_S(a) \cup f_S(b). \end{aligned}$$

Thus, (f_S, S) is a uni-soft semigroup over U.

By Propositions 3 and 5, we have the following:

Remark.*In regular ordered semigroups the concepts of uni-soft generalized bi-ideals and uni-soft bi-ideals coincide.*

Proposition 6.*Let (f_S, S) be a uni-soft generalized bi-ideal of a regular ordered semigroup over U. Then for all $a \in S$ such that $a \leq a^2$, we have*

$$f_S(a) = f_S(a^2).$$

Proof. Sppose that (f_S, S) be a uni-soft generalized bi-ideal over U. Let $a \in S$ be such that $a \leq a^2$. Then

$$\begin{aligned} f_S(a) &\subseteq f_S(a^2) = f_S(aa) \\ &\subseteq f_S(a) \cup f_S(a) \\ &= f_S(a). \end{aligned}$$

Hence $f_S(a) = f_S(a^2)$.

Theorem 3. For an ordered semigroup S, the following conditions are equivalent:

(1) S is regular.

(2) $(f_S, S) = (f_S \Diamond \phi_S \Diamond f_S, S)$ for every uni-soft generalized bi-ideal (f_S, S) over U.

Proof. First assume that (1) holds. Let (f_S, S) be a uni-soft generalized bi-ideal over U and $a \in S$. Since S is regular, then there exists $x \in S$ such that $a \leq axa$. Thus, we have

$$\begin{aligned}(f_S \Diamond \phi_S \Diamond f_S)(a) &= \bigcap_{a \leq pq} \{(f_S \Diamond \phi_S)(p) \cup f_S(q)\} \\ &\subseteq (f_S \Diamond \phi_S)(ax) \cup f_S(a) \\ &= \bigcap_{ax \leq uv} \{f_S(u) \cup \phi_S(v)\} \cup f_S(a) \\ &\subseteq f_S(a) \cup \phi_S(x) \cup f_S(a) \\ &= f_S(a) \cup \phi \cup f_S(a) \\ &= f_S(a).\end{aligned}$$

Which means that $(f_S \Diamond \phi_S \Diamond f_S, S) \tilde{\subseteq} (f_S, S)$ and by theorem 1, we have $(f_S \Diamond \phi_S \Diamond f_S, S) \tilde{\supseteq} (f_S, S)$ Thus, $(f_S, S) = (f_S \Diamond \phi_S \Diamond f_S, S)$.

Conversely, assume that (2) holds. To prove that S is regular, we need to illustrate that $X \subseteq (XSX]$. For this let $a \in X$. Then, by Proposition 2, the soft set (χ_X^c, S) is a uni-soft generalized bi-ideal of S over U. Thus, we have

$$\begin{aligned}(\chi^c_{(XSX]})(a) &= (\chi^c_X \Diamond \chi^c_S \Diamond \chi^c_X)(a) \\ &= (\chi^c_X \Diamond \phi \Diamond \chi^c_X)(a) \\ &= (\chi^c_X)(a) \\ &= \phi,\end{aligned}$$

which means that $a \in (XSX]$. Thus, $X \subseteq (XSX]$. It follows that S is regular.

Lemma 5.[21]For an ordered semigroup, the following conditions are equivalent:

(1) S is regular.

(2) $B \cap L \subseteq (BL]$, for every generalized bi-ideal B and left ideal L of S.

(3) $B(x) \cap L(x) \subseteq (B(x)L(x)], \forall x \in S$.

Theorem 4.*For an ordered semigroup, the following conditions are equivalent:*

(1) S is regular.

(2) $(f_S \tilde{\cup} g_S, S) \tilde{\supseteq} (f_S \Diamond g_S, S)$ for every uni-soft generalized bi-ideal (f_S, S) and uni-soft left ideal (g_S, S) over U.

Proof .First Assume that (1) holds. Let (f_S, S) be a uni-soft generalized bi-ideal and (g_S, S) be a uni-soft left ideal over U. Let a be an element in S, since S is regular. Then there exists $x \in S$, such that $a \leq axa \leq axaxa$. Then $(axa, xa) \in A_a$. Since $A_a \neq \phi$, we have

$$\begin{aligned}(f_S \Diamond g_S)(a) &= \bigcap_{a \leq yz} \{f_S(y) \cup g_S(z)\} \\ &\subseteq \{f_S(axa) \cup g_S(xa)\}.\end{aligned}$$

Since (f_S, S) is a uni-soft generalized bi-ideal over U, we have $f_S(axa) \subseteq f_S(a) \cup f_S(a) = f_S(a)$ and (g_S, S) is a uni-soft left ideal over U, we have $g_S(xa) \subseteq g_S(a)$. Thus

$$\begin{aligned}(f_S \Diamond g_S)(a) &\subseteq \{f_S(axa) \cup g_S(xa)\}. \\ &\subseteq f_S(a) \cup g_S(a) \\ &= (f_S \tilde{\cup} g_S)(a).\end{aligned}$$

Therefore, $(f_S \tilde{\cup} g_S, S) \tilde{\supseteq} (f_S \Diamond g_S, S)$.

Conversely, assume tha (2) holds. To prove that S is regular, by Lemma 5, it is enough to prove that

$$B(x)\cap L(x)\subseteq (B(x)L(x)],\forall x\in S.$$

Let $y\in B(x)\cap L(x)$. Then $y\in (B(x)L(x)]$. Since $B(x)$ is the generalized bi-ideal and $L(x)$ is the left ideal of S generated by x, respectively. Then by Lemma 3, $\left(\chi^c_{\mathbf{L}(x)},S\right)$ is a uni-soft left ideal and by Proposition 2, $\left(\chi^c_{\mathbf{B}(x)},S\right)$ a uni-soft generalized bi-ideal over U. Then by Proposition 1 and hypothesis, we have

$$\chi^c_{(\mathbf{B}(x)\mathbf{L}(x)]}(y)=(\chi^c_{B(x)}\Diamond\chi^c_{L(x)})(y)\subseteq(\chi^c_{B(x)}\tilde{\cup}\chi^c_{L(x)})(y)=\chi_{B(x)}(y)\cup\chi^c_{L(x)}(y)=\phi,$$

which implies that $y\in\left(B(x)L(x)\right]$. Thus $B(x)\cap L(x)\subseteq\left(B(x)L(x)\right]$. It follows from Lemma 5, that is S is regular.

Proposition 7. Let S be an ordered semigroup, $\left(f_S,S\right)$ is a uni-soft generalized bi-ideal and $\left(g_S,S\right)$ is a uni-soft ideal over U. Then we have

$$(f_S\Diamond g_S\Diamond f_S,S)\tilde{\supseteq}\left(f_S\tilde{\cup}g_S,S\right).$$

Proof.Let $\left(f_S,S\right)$ be a uni-soft generalized bi-ideal and $\left(g_S,S\right)$ be a uni-soft ideal over U. Let $a\in S$. Then $(f_S\Diamond g_S\Diamond f_S,S)\tilde{\supseteq}\left(f_S\tilde{\cup}g_S,S\right)$. In fact: If $A_a=\phi$, then $(f_S\Diamond g_S\Diamond f_S)(a)=U\supseteq f_S(a)\cup g_S(a)=\left(f_S\tilde{\cup}g_S,S\right)$.Let $A_a\neq\phi$, then

$$\begin{aligned}(f_S\Diamond g_S\Diamond f_S)(a)&=\bigcap_{a\leq yz}\{f_S(y)\cup(g_S\Diamond f_S)(z)\}\\&=\bigcap_{a\leq yz}\left\{f_S(y)\cup\left(\bigcap_{z\leq pq}\{g_S(p)\cup f_S(q)\}\right)\right\}\\&=\bigcap_{a\leq yz}\bigcap_{z\leq pq}\{f_S(y)\cup g_S(p)\cup f_S(q)\},\end{aligned}\tag{1}$$

For each $(y,z)\in A_a$ and $(p,q)\in A_z$, we have

$$a\leq yz \text{ and } z\leq pq. \text{Thus } a\leq ypq.$$

Since $\left(f_S,S\right)$ is a uni-soft generalized bi-ideal over U, so we have

$$f_S(a)\subseteq f_S(ypq)\subseteq f_S(y)\cup f_S(q)$$

And $\left(g_S,S\right)$ is a uni-soft ideal over U, so we have

$$g_S(a)\subseteq g_S(ypq)\subseteq g_S(p).$$

Thus

$$\begin{aligned}f_S(a)\cup g_S(a)&\subseteq\{f_S(y)\cup f_S(q)\}\cup g_S(p)\\&=f_S(y)\cup f_S(p)\cup g_S(q).\end{aligned}$$

Hence from (1) we get

$$(f_S\Diamond g_S\Diamond f_S)(a)\supseteq f_S(a)\cup g_S(a)=(f_S\tilde{\cup}g_S)(a).$$

Thus, $(f_S\Diamond g_S\Diamond f_S,S)\tilde{\supseteq}\left(f_S\tilde{\cup}g_S,S\right)$.

Lemma6.[21] *For an ordered semigroup, the following conditions are equivalent:*

(1) S is regular.

(2) $B\cap I=(BIB]$, for every generalized bi-ideal B and ideal I of S.

(3) $B(x)\cap I(x)=(B(x)I(x)B(x)],\forall x\in S$.

Theorem 5. *For an ordered semigroup, the following conditions are equivalent:*

(1) S is regular.

(2) $(f_S\Diamond g_S\Diamond f_S,S)=\left(f_S\tilde{\cup}g_S,S\right)$. for every uni-soft generalized bi-ideal $\left(f_S,S\right)$ and every uni-soft ideal $\left(g_S,S\right)$ over U.

Proof.Assume that (1) holds. Let $\left(f_S,S\right)$ be a uni-soft generalized bi-ideal and $\left(g_S,S\right)$ be a uni-soft ideal over U. Since S is regular, then for each element a in S, there exists $x\in S$, such that

$$a \le axa \le (axa)xa.$$

Thus, $(axa, xa) \in A_a$. Hence

$$\begin{aligned}(f_S \Diamond g_S \Diamond f_S)(a) &= \bigcap_{a \le yz} \{f_S(y) \cup (g_S \Diamond f_S)(z)\} \\ &\subseteq f_S(axa) \cup (g_S \Diamond f_S)(xa) \\ &= f_S(axa) \cup \left\{ \bigcap_{xa \le pq} \{g_S(p) \cup f_S(q)\} \right\} \\ &\subseteq f_S(axa) \cup \{g_S(xax) \cup f_S(axa)\} \text{ since } xa \le x(axa) \le (xax)(axa) \\ &= f_S(axa) \cup g_S(xax).\end{aligned}$$

As (f_S, S) is a uni-soft generalized bi-ideal and (g_S, S) is a uni-soft ideal over U, so we have $f_S(axa) \subseteq f_S(a) \cup f_S(a) = f_S(a)$, $g_S(xax) \subseteq g_S(a)$. Thus

$$f_S(axa) \cup g_S(xax) \subseteq f_S(a) \cup g_S(a) = (f_S \tilde{\cup} g_S)(a).$$

Therefore, $(f_S \Diamond g_S \Diamond f_S, S) \tilde{\subseteq} (f_S \tilde{\cup} g_S, S)$.

On the other hand, by Proposition 7, we have $(f_S \Diamond g_S \Diamond f_S, S) \tilde{\supseteq} (f_S \tilde{\cup} g_S, S)$. Thus

$$(f_S \Diamond g_S \Diamond f_S, S) = (f_S \tilde{\cup} g_S, S).$$

Conversely, assume that (2) holds. To prove that S is regular, by Lemma 6, it is enough to show that

$$B(x) \cap I(x) = (B(x)I(x)B(x)], \forall x \in S.$$

Let $y \in B(x) \cap I(x)$. Then $y \in (B(x)I(x)B(x)]$. Since $B(x)$ is the generalized bi-ideal and $I(x)$ is the ideal of S, generated by x, respectively. Then by Lemma 3, $(\chi^c_{I(x)}, S)$ is a uni-soft ideal and by Proposition 2, $(\chi^c_{B(x)}, S)$ is a uni-soft generalized bi-ideal over U. Then by Proposition 1 and hypothesis , we have

$$\begin{aligned}(\chi^c_{(B(x)I(x)B(x)]})(y) &= (\chi^c_{B(x)} \Diamond \chi^c_{I(x)} \Diamond \chi^c_{B(x)})(y) \\ &= (\chi^c_{B(x)} \tilde{\cup} \chi^c_{I(x)})(y) \\ &= \chi^c_{B(x)}(y) \cup \chi^c_{I(x)}(y) \\ &= \varphi,\end{aligned}$$

which means that $y \in (B(x)I(x)B(x)]$. Thus $B(x) \cap I(x) \subseteq (B(x)I(x)B(x)]$. It follows Lemma 6, that is S is regular.

Lemma 7.[21] *For an ordered semigroup, the following conditions are equivalent:*

(1) S is regular.

(2) $R \cap B \cap L \subseteq (RBL]$, for every right ideal R, generalized bi-ideal B and every left ideal L of S.

(3) $R(x) \cap B(x) \cap L(x) \subseteq (R(x)B(x)L(x)], \forall x \in S$.

Theorem 5. For an ordered semigroup, the following conditions are equivalent:

(1) S is regular.

(2) $(f_S \tilde{\cup} g_S \tilde{\cup} h_S, S) \tilde{\supseteq} (f_S \Diamond g_S \Diamond h_S, S)$ for every uni-soft right ideal (f_S, S), every uni-soft generalized bi-ideal (g_S, S) and every uni-soft left ideal (h_S, S) over U.

Proof. Firstassumes that (1) holds. Let $(f_S, S), (g_S, S)$ and (h_S, S) be uni-soft right, uni-soft generalized bi-ideal and uni-soft left ideal over U respectively. Since S is regular, then for every $a \in S$, there exists $x \in S$, such that $a \le axa$. Hence $(ax, a) \in A_a$. Since $A_a \ne \phi$, we have

$$\begin{aligned}(f_S \Diamond g_S \Diamond h_S)(a) &= \bigcap_{a \le yz} \{f_S(y) \cup (g_S \cup h_S)(z)\} \\ &\subseteq \{f_S(ax) \cup (g_S \cup h_S)(a)\} \\ &= f_S(ax) \cup \left\{ \bigcap_{a \le pq} \{g_S(p) \cup h_S(q)\} \right\} \\ &\subseteq f_S(ax) \cup \{g_S(axa) \cup h_S(xa)\} \ (\text{since } a \le axa \le (axa)xa) \\ &= f_S(ax) \cup g_S(axa) \cup h_S(xa).\end{aligned}$$

As (f_S, S) is a uni-soft right ideal, (g_S, S) is a uni-soft generalized bi-ideal and (h_S, S) is a uni-soft left ideal over U, we have $f_S(ax) \subseteq f_S(a)$, $g_S(axa) \subseteq g_S(a) \cup g_S(a) = g_S(a)$ and $h_S(xa) \subseteq h_S(a)$. Thus

$$f_S(ax) \cup g_S(axa) \cup h_S(xa) \subseteq f_S(a) \cup g_S(a) \cup h_S(a) = (f_S \tilde{\cup} g_S \tilde{\cup} h_S)(a).$$

Therefore, $(f_S \tilde{\cup} g_S \tilde{\cup} h_S, S) \tilde{\supseteq} (f_S \Diamond g_S \Diamond h_S, S)$.

Conversely, assume that (2) holds. To prove that S is regular, by Lemma 7, it is enough to prove that

$$R(x) \cap B(x) \cap L(x) \subseteq (R(x)B(x)L(x)], \text{ for every } x \in S.$$

Let $y \in R(x) \cap R(x) \cap L(x)$. Then $y \in (R(x)B(x)L(x)]$. Since $R(x)$, $B(x)$ and $L(x)$ be right ideal, generalized bi-ideal and left ideal of S generated by x, respectively. Therefore by Lemma 3 and Proposition 2, $(\chi^c_{R(x)}, S)$, $(\chi^c_{L(x)}, S)$ and $(\chi^c_{B(x)}, S)$ are uni-soft right ideal, uni-soft left ideal and uni-soft generalized bi-ideal over U respectively. Then by Proposition 1 and hypothesis, we have

$$\begin{aligned}(\chi^c_{(R(x)B(x)L(x)]})(y) &= (\chi^c_{R(x)} \Diamond \chi^c_{B(x)} \Diamond \chi^c_{L(x)})(y) \\ &= (\chi^c_{R(x)} \tilde{\cup} \chi^c_{B(x)} \tilde{\cup} \chi^c_{L(x)})(y) \\ &= \chi^c_{R(x)}(y) \cup \chi^c_{B(x)}(y) \cup \chi^c_{L(x)}(y) \\ &= \phi,\end{aligned}$$

which means that $y \in (R(x)B(x)L(x)]$. Thus $R(x) \cap B(x) \cap L(x) \subseteq (R(x)B(x)L(x)]$. It follows Lemma 7, that is S is regular.

Uni-soft generalized bi-ideals of completely regular and left weakly regular ordered semigroups

In this section, we characterize completely regular and left weakly regular ordered semigroups in terms of uni-soft generalized bi-ideals. We also study different properties of uni-soft generalized bi-ideals as a regards of uni-soft product. An element a of S is called a completely regular if there exists $x \in S$ such that

$$a \le axa \text{ and } ax = xa$$

An ordered semigroup S is called completely regular if every element of S is completely regular. An ordered semigroup S is called *left* $(right)$ *weakly-regular* if for every element a in S, there exists $x, y \in S$ such that $a \le xaya (a \le axay)$ *[8, 9]*.

Proposition 8.*Let* S *be an ordered semigroup such that*

(i) $(\forall x \in S)(x \le x^2)$,

(ii) $(\forall a, b \in S)(ab \in (baS] \cap (Sba])$.

Then every uni-soft generalized bi-ideal (f_S, S) over U satisfies the following condition

$$f_S(ab) = f_S(ba), \forall a, b \in S.$$

Proof. Since $ab \in (baS] \cap (Sba]$, then $ab \in (baS]$ and so, $ab \le bax$ for some $x \in S$. using (ii), we get $(ba)x \in (xbaS] \cap (Sxba]$ and thus $bax \le yxba$ far some $y \in S$. It follows from (i) that $ab \le (ba)x \le (ba)^2 x = (ba)(bax) \le (ba)(yxba)$. Since (f_S, S) uni-soft generalized bi-ideal over U, we have

$$\begin{aligned} f_S(ab) &\subseteq f_S(ba(yxba)) = f_S(ba(yx)ba) \\ &\subseteq f_S(ba) \cup f_S(ba) \\ &= f_S(ba). \end{aligned}$$

By symetry we can prove that $f_S(ba) \subseteq f_S(ab)$. Hence $f_S(ab) = f_S(ba)$.

Theorem 7. If every uni-soft generalized bi-ideal (f_S, S) over U satisfies $f_S(a) = f_S(a^2)$ for all $a \in S$, then S is completely regular.

Proof. Assume that every uni-soft generalized bi-ideal (f_S, S) over U satisfies $f(a) = f(a^2)$ for all $a \in S$ and let $x \in S$. Note that $\mathbf{B}(x^2) = (x^2 \cup x^2Sx^2]$ is the generalized bi-ideal of S generated by x^2, by Proposition 2, $(\chi^c_{(\mathbf{B}(x^2)]}, S)$ is a uni-soft generalized bi-ideal over U. By hypothesis, $\chi^c_{(\mathbf{B}(x^2)]}(x) = \chi^c_{(\mathbf{B}(x^2)]}(x^2)$. Since $x^2 \in (\mathbf{B}(x^2)]$ we have $\chi^c_{(\mathbf{B}(x^2)]}(x^2) = \phi$ and so $\chi^c_{(\mathbf{B}(x^2)]}(x) = \phi$ Thus $x \in (\mathbf{B}(x^2)] := (x^2 \cup x^2Sx^2]$, which implies that $x \leq a$ for some $a \in x^2 \cup x^2Sx^2$. If $a = x^2$, then

$$x \leq a = x^2 = xx \leq x^2x^2 = xxx^2 \leq x^2xx^2,$$

and so $x \in (x^2Sx^2]$. If $a \in x^2Sx^2$ then obviously $x \in (x^2Sx^2]$. Hence S is completely regular.

Proposition 9.[21]In a left weakly regular semigroup S, every generalized bi-ideal is a subsemigroup, that is, a bi-ideal.

Proposition 10. In a left weakly regular semigroup S, every uni-soft generalized bi-ideal over U is a uni-soft semigroup, that is, a uni-soft bi-ideal over U.

Proof. Let (f_S, S) be a uni-soft generalized bi-ideal over U. Let $a, b \in S$, since S is left weakly regular, then there exists $x, y \in S$, such that

$$b \leq xbyb \Rightarrow ab \leq a(xbyb) = a(xby)b.$$

Since (f_S, S) is a uni-soft generalized bi-ideal over U, we have

$$f_S(ab) \subseteq f_S(a(xby)b) \subseteq f_S(a) \cup f_S(b).$$

It follows that (f_S, S) is a uni-soft semigroup over U.

By Propositions 3 and 11, we have the following:

Remark 2. In a left weakly regular ordered semigroup, the concepts of uni-soft generalized bi-ideals and uni-soft bi-ideals coincide.

Lemma 8.[21]For an ordered semigroup, the following conditions are equivalent:

(1) S is left weakly regular.

(2) $B \cap I \subseteq (IB]$, for every generalized bi-ideal B and ideal I of S.

(3) $B(x) \cap I(x) \subseteq (I(x)B(x)], \forall x \in S$.

Theorem 8.For an ordered semigroup, the following conditions are equivalent:

(1) S is left weakly regular.

(2) $(f_S \tilde{\cup} g_S, S) \tilde{\supseteq} (f_S \Diamond g_S, S)$, for every uni-soft generalized bi-ideal (f_S, S) and uni-soft ideal (g_S, S) over U.

Proof.Assume that (1) holds. Let (f_S, S) and (g_S, S) be a uni-soft generalized bi-ideal and uni-soft ideal over U respectively. Since S is left weakly regular. Then for each element a in S, there exists $x, y \in S$, such that

$$a \leq xaya \leq xay(xaya) = (xayx)(aya).$$

Thus $(xayx, aya) \in A_a$. Since $A_a \neq \phi$, we have

$$\begin{aligned} (g_S \Diamond f_S)(a) &= \bigcap_{a \leq st} \{g_S(s) \cup f_S(t)\} \\ &\subseteq g_S(xayx) \cup f_S(aya). \end{aligned}$$

Since (f_S, S) is a uni-soft generalized bi-ideal and (g_S, S) is a uni-soft ideal over U, we have $f_S(aya) \subseteq f_S(a) \cup f_S(a) = f_S(a)$ and $g_S(xayx) = g_S(xa(yx)) \subseteq g_S(a)$.

$$(g_S \Diamond f_S)(a) \subseteq g_S(xayx) \cup f_S(aya) \subseteq g_S(a) \cup f_S(a) = (f_S \tilde{\cup} g_S)(a).$$

Therefore, $(f_S \tilde{\cup} g_S, S) \tilde{\supseteq} (f_S \Diamond g_S, S)$.

Conversely, assume that (2) holds. To prove that S is left weakly regular, by Lemma 8, it is enough to prove that

$$B(x) \cap I(x) \subseteq (I(x)B(x)], \forall x \in S.$$

Let $y \in B(x) \cap I(x)$, then $y \in (I(x)B(x)]$. Since $I(x)$ and $B(x)$ be ideal and generalized bi-ideal of S generated by x, respectively. Then by Lemma 3 and Propositions 2, $(\chi^c_{I(x)}, S)$ is a uni-soft ideal and $(\chi^c_{B(x)}, S)$ a uni-soft generalized bi-ideal over U respectively. By Proposition 1 and hypothesis, we have

$$\begin{aligned}(\chi^c_{(I(x)B(x)]})(y) &= (\chi^c_{I(x)} \Diamond \chi^c_{B(x)})(y)\\ &= (\chi^c_{I(x)} \tilde{\cup} \chi^c_{B(x)})(y)\\ &= \chi^c_{I(x)}(y) \cup \chi^c_{B(x)}(y)\\ &= \phi,\end{aligned}$$

which means that $y \in (I(x)B(x)]$. Thus $B(x) \cap I(x) \subseteq (I(x)B(x)]$. It follows Lemma 8, that is S is regular.

REFERENCES

[1] M. I.Ali, F. Feng, X. Liu, W. K.Min, M.Shabir.,. On some new operations in soft set theory. Comput.Math. Appl. $57\ (2009)$ (9)$1547-1553$.

[2] N. Cagman and S. Enginoglu, Soft set theory and uni--int decision making, European Journal of Operational Research, vol. 207, no. 2, pp. 848--855, 2010.3.

[3] F. Feng, M. I. Ali, and M. Shabir, Soft relations applied to semigroups, Filomat, vol. 27, no. 7, pp. 1183--1196, 2013.

[4] F. Feng, Y. B. Jun and X. Zhao, "Soft semirings," Comput. Math. Appl. 56 (2008), 2621-2628.

[5] Y. B. Jun, S. Z. Song, and G. Muhiuddin, "Concave soft sets, critical soft points, and union-soft ideals of ordered semigroups," The Scientific World Journal, Volume 2014, Article ID 467968, 11 pages.

[6] Y. B. Jun, K. J. Lee, and A. Khan, Soft ordered semigroups, Mathematical Logic Quarterly, vol. 56, no. 1, pp. 42--50, 2010.

[7] N. Kehayopulu, "On regular ordered semigroups," Math. Japonica 45 (3), (1997), 549-553.

[8] N. Kehayopulu and M. Tsingelis, "Regular ordered semigroups in terms of fuzzy subsets," Inform. Sci. 176 (2006), no. 24, 3675-3693.

[9] N. Kehayopulu, "On completely regular ordered semigroups," Sci. Math. 1 (1998), no. 1, 27-32 (electronic).

[10] N. Kehayopulu, "Remark on ordered semigroups," Math. Japon. 35 (1990), no. 6, 1061-1063.

[11] A. Khan, Y. B. Jun and M. Shabir, "Fuzzy ideals in ordered semigroups-I", Quasigroups Related Systems 16 (2008), no. 2, 207-220.

[12] A. Khan, Y. B. Jun and R. Khan, "Characterizations of ordered semigroups in terms of union-soft ideals," submitted.

[13] A. Khan, Y. B. Jun, S. I. A, Shah and R. Khan, "Applications of soft-union sets in ordered semigroups via SU quasi-ideals," in press.

[14] A. Khan, R. Khan, and Y. B. Jun, "Uni-soft structure applied to ordered semigroups," Soft Comput. DOI 10.1007/s00500-015-1837-8.

[15] P. K. Maji, R. Biswas and A. R. Roy, "Soft set theory," Comput. Math. Appl., 45 (2003), 555-562.

[16] P. K. Maji, R. Biswas and A. R. Roy, "Fuzzy Soft Sets," Journal of Fuzzy Mathematics, 9(2001), 589-602.

[17] P. K. Maji, A. R. Roy and R. Biswas, "An application of soft sets in a decision making problem," Comput. Math. Appl., 44 (2002), 1077-1083.

[18] D. Molodtsov, Soft set theory---first results, Computers & Mathematics with Applications, vol. 37, no. 4-5, pp. 19--31, 1999.

[19] Aktas : H. Aktas and N. Cağman, "Soft sets and soft groups," Information Sciences, vol. 177, no. 13, pp. 2726--2735, 2007.

[20] V.S. Ramamurthy, "Weakly regular rings," Canad. Math. Bull. 18 (1973) 317-321.

[21] M. Shabir and M.I. Ali, "Soft ideals and generalized fuzzy ideals in semigroups," New Mathematics and Natural Computation, 5(3)(2009), 599-615.

[22] M. Shabir and A. Khan, "Characterizations of ordered semigroups by the properties of their fuzzy generalized bi-ideals," New Mathematics and Natural Computation. Vol. 4, No. 2 (2008) 237-250.

[23] M. Shabir and A. Khan, "Characterizations of ordered semigroups by the properties of their fuzzy ideals," Comput. Math. Appl. 59 (2010) 539-549.

[24] L. A. Zadeh, "Fuzzy sets," Information and Control 8 (1965), 338-35.

[25] J. Zhan, N. Čağman and A. S. Sezer, "Applications of soft union sets to hemiringsvia SU - h -ideals," J. Intel. Fuzzy Syst. 26 (2014) 1363-1370.

[26] J. Zhan and Y. Yin, "A new view of fuzzy k-ideals of hemirings," J Intell Fuzzy Syst 23 (2012), 169-176.