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Numerical Solutions of Some Higher Order Fractional Integro-Differential Equations (FIDEs), Using Chebyshev Wavelet Method (CWM)

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ABSTRACT

In this paper, the numerical solutions for some of the Integro-differential Equations are obtained, using Chebyshev Wavelet Method (CWM). The numerical simulations for integer and fractional orders are handled in a very efficient, simple and straight forward procedure. The numerical results of the proposed method are compared with the exact solutions of the problems. It is observed, that the solutions obtained by the present method have a strong agreement with the exact solutions of the problems. The solutions by this method have high degree of accuracy than any other numerical method. The numerical solutions of fractional order Integro-Differential Equations, shows convergence toward the integer order solutions.
KEYWORDS: Integro differential equations, Chebyshev Wavelet Method

INTRODUCTION

Integral and Integro-differential equations have a vital role in deterministic models of social, biological, physical and engineering sciences [1, 2]. The analytical and exact solutions of nonlinear Integro-differential equations are usually rather difficult and hard to be obtained. In this regard many numerical methods have been developed and studied such as Legendre wavelet method [3], Finte difference method [4], Adomian decomposition method [5], Haar function method [6] and Taylor polynomial method [7].

Recently fractional calculus has taken much importance due its useful applications in science and technology. In this context Gemant have started the study of fractional calculus [8]. Other applications are Nonlinear Oscillation of earth quake [9], Signal Processing [10], Control Theory [11] and Fluid Dynamics Traffic [12]. The above physical problems consist of mathematical models based on fractional calculus.

The exact and numerical solution to the Fractional Order Differential Equations is the focus point of many researchers in the past Therefore a number of powerful and efficient methods have been suggested to obtain approximate solution of Fractional Differential Equations. These methods are Adomian Decomposition Method (ADM) [13], The Variation Iteration Method (VIM) [14], Homotopy Analysis Method (HAM) [15], The Homotopy Perturbation Method (HPM) [16], and the Ex-Function Method (EFM) [17].

Recently most of the researchers have taken great interest in Wavelet Theory [18- 24]. This theory is based on some relatively new techniques based on Wavelets. The gradual improvement in this method has been observed, which are helpful in increasing the accuracy of the present method. The most relevant methods are Haar Wavelets [22], Legendre Wavelets [21], Harmonic wavelet method [19], CAS wavelets [21] and Chebyshev Wavelets [22-25].

In the current work, we have applied a numerical method based on Chebyshev Wavelet of second kind for the numerical solutions of some fractional higher order differential equations. The numerical simulations are done easily. The solutions are compared with the solutions obtain by MOHAM and also with the exact solutions. The numerical results have shown that the proposed method has the highest degree of accuracy over all the methods under discussion.

PRELIMINARIES AND DEFINITIONS

In the connection of the current work, some of the necessary mathematical preliminaries and definitions are extremely important to work with fractional calculus.

Definition1. The Riemann fractional integral operator I^{γ} of order γ on the usual Lebesgue space $L_1[a,b]$ is given by
$(I^{\gamma}g)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}g(\xi)d\xi,\ \gamma > 0,$
with $(I^0 g)(t) = g(t).$

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The above Integral operator has the following properties

$$(i)\ I^{\gamma}I^{\eta} = I^{\gamma+\eta},\ \text{(ii)}\ I^{\gamma}I^{\eta} = I^{\eta}I^{\gamma},\ \text{(iii)}\ I^{\gamma}(t-a)^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\alpha+\upsilon+1)}(t-a)^{\alpha+\upsilon},$$

where

$$L_1[a,b], \alpha, \gamma \geq 0, \text{ and } \upsilon > -1.$$

Definition2. The Riemann fractional derivative of order $\gamma > 0$ is defined as

$$(D^{\gamma}g)(t) = \left(\frac{d}{dt}\right)^{n}(I^{n-\gamma}g)(t), n-1 < \gamma \leq n$$

where, n is an integer.

However the Riemann fractional derivative has certain drawbacks due to which Caputo proposed a modified differential operator**.**

Definition3. The Caputo definition of fractional differential operator is given by

$$(D^{\gamma}g)(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t (t-\xi)^{n-\gamma-1}g^{(n)}(\xi)d\xi,\ n-1 < \gamma > n,$$

where $t > 0$, n is an integer

It has the following two basic properties

$$(i)\ (D^{\gamma}I^{\gamma}g)(t) = g(t),$$

$$\text{(ii)}\ (I^{\gamma}D^{\gamma}g)(t) = (g)(t) - (x+a)^n = \sum_{k=0}^{n} f^{(k)}(0^+)\frac{(t-a)^k}{k!}, t > 0.$$

Properties of the Chebyshev Wavelets

Wavelets consist of family of functions generated from the dilation a and translation b of a single function $\psi(x)$ called the mother wavelet. When the dilation a and translation b change continuously then we get the following continuous family of wavelets [21]:

$$\psi_{a,b}(x) = |a|^{\frac{1}{2}}\psi\left(\frac{x-b}{a}\right), a, b \in R, a \neq 0,$$

If we restrict the parameters a and b to discrete values then it takes the form:

$$a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 1.$$

We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{\frac{k}{2}}\psi(a_0^{k}x - nb_0), k, n \in Z,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. Especially when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ form an orthogonal basis.

The second kind of Chebyshev wavelets is constituted of four parameters,$\psi_{n,m}(x) = \psi(k,n,m,x)$, where $n = 1,2, ...,2^{k-1}$, k is any positive integer, m is the degree of the second Chebyshev polynomials. The Chebyshev wavelets are defined on the interval 0≤ $x < 1$ as

$$\psi_{n,m}(x) = \begin{cases} 2^{k/2}\tilde{T}(2^k - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & 0therwise \end{cases} \quad (1)$$

where

$$\widetilde{T_m}(x) = \sqrt{\frac{2}{\pi}}T_m(x), m = 0,1,2, ..., M-1. \quad (2)$$

Here $T_m(x)$ are the second Chebyshev polynomials of degree m with respect to the weight function $w(x) = \sqrt{1-x^2}$ on the interval [-1,1], and satisfying the following recursive formula

$$T_0(x) = 1, T_1(x) = 2x,$$

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), m = 1,2,3, ...,$$

Chebyshev Wavelet Method (CWM): In this paper we consider the fractional integro-differential equations of the form

$$\sum_{i=0}^{m}\mu_i(x)y^{(i)}(x) = f(x) + \lambda_1\int_0^x k_1(x,t)\big(y(t)\big)^p dt + \lambda_2\int_a^b k_2(x,t)\big(y(t)\big)^q dt = 0, \quad (3)$$

with the initial conditions given by

$$y^{(i)}(a) = y_i,$$

for $i = 1,2,3,........m-1$, λ_1, λ_2 and y_i are constants and $p, q \in$ Z. $f(x), k_1(x,t), k_2(x,t), y(x)$ and $\mu_i(x)$ are known functions of x. These functions are differentiable in the interval $a \leq x \leq t \leq b$.

The solution to equation (3) can be extended by Chebyshev wavelets series as

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\,\psi_{n,m}(x) \qquad (4)$$

Where $\psi_{n,m}(x)$ is given by equation (1). The series in equation (4) is truncated to finite number of terms that is

$$y_{k,M}(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(x) \qquad (5)$$

This shows that there are $2^{k-1}M$ conditions to determine $2^{k-1}M$ coefficients, $c_{i,j}$

In the present work we have considered some initial and boundary value, Integro differential equations of order three and four. We have four boundary conditions for Integro differential equation of order four, therefore four conditions are obtained by these boundary conditions. These conditions are

$$y_{k,M}(0) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(0) = \alpha_0, \qquad (6)$$

$$y_{k,M}(1) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(0) = \alpha_{1,} \qquad (7)$$

$$\frac{d}{dx}y_{k,M}(0) = \frac{d}{dx}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(x)\right) = \beta_{0,} \qquad (8)$$

$$\frac{d}{dx}y_{k,M}(1) = \frac{d}{dx}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(x)\right) = \beta_1. \qquad (9)$$

The remaining $2^{k-1}M-4$ conditions can be obtain by substituting equation (5) in equation (3), we get

$$\frac{d^\alpha}{dx^\alpha}\sum_{i=0}^{m}\mu_i(x)\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5} c_{n,m}\,\psi_{n,m}(x_i) = f(x) + \lambda_1\int_0^x k_1(x,t)\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-5} c_{n,m}\,\psi_{n,m}(x_i)\right)^p dt +$$

$$\lambda_2\int_a^b k_2(x,t)\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-11} c_{n,m}\,\psi_{n,m}(x_i)\right)^q dt = 0. \qquad (10)$$

Assume that equation (6) is exact at $2^{k-1}M-4$ points, which we call it x_i, then

The x_i points are obtained by using the following formula

$x_i = \frac{i-0.5}{2^{k-1}M}$, for $i = 1,2,3,\ldots 2^{k-1}M-4$.

The combination of equations (6), (7), (8), (9) and (10) form the linear system of $2^{k-1}M$ linear equations. To determine the unknown coefficients $c_{i,j}$ we will solve this linear system of equations. The same procedure can be repeated for other fractional integro-differential equations of various orders.

Numerical Examples

Example 1:

Consider the following fractional order nonlinear boundary value problem [26]

$$\frac{d^\alpha y}{dx^\alpha} - x(1+e^x) - 3e^x - y + \int_0^x u(t)dt = 0, 3 \le \alpha \le 4, 0 \le x \le 1, \qquad (11)$$

with boundary conditions

$$y(0) = 1,\ y(1) = 1+e, y''(0) = 2,\ \ y''(1) = 3e.$$

The exact solution is

$$y(x) = 1 + xe^x.$$

Table 1: Numerical results of Example 1 for $\alpha = 4$

x	y_{exact}	y_4	Errory_4	Error MOHPM
0.0	1.00000000000000000000	1.000000000000000000	0.000000	0.00000
0.1	1.11051709180756485070	1.11051709180756476250	8.82E-17	-4.00E-11
0.2	1.24428055163203411630	1.24428055163203396680	1.49E-16	-7.90E-11
0.3	1.40495764227280111690	1.40495764227280093120	1.85E-16	-1.10E-10
0.4	1.59672987905650832740	1.59672987905650812710	2.00E-16	-1.40E-10
0.5	1.82436063535006426890	1.82436063535006407340	1.95E-16	-1.60E-10
0.6	2.09327128023430556010	2.09327128023430538490	1.75E-16	-1.7E-10
0.7	2.40962689522933370750	2.40962689522933356510	1.42E-16	-1.50E-10
0.8	2.78043274279397418370	2.78043274279397408370	1.00E-16	-1.1E-10
0.9	3.21364280004125474900	3.21364280004125469740	5.16E-17	-5.90E-11
1.0	3.71828182845904523520	3.71828182845904523540	2.00E-19	1.20E-11

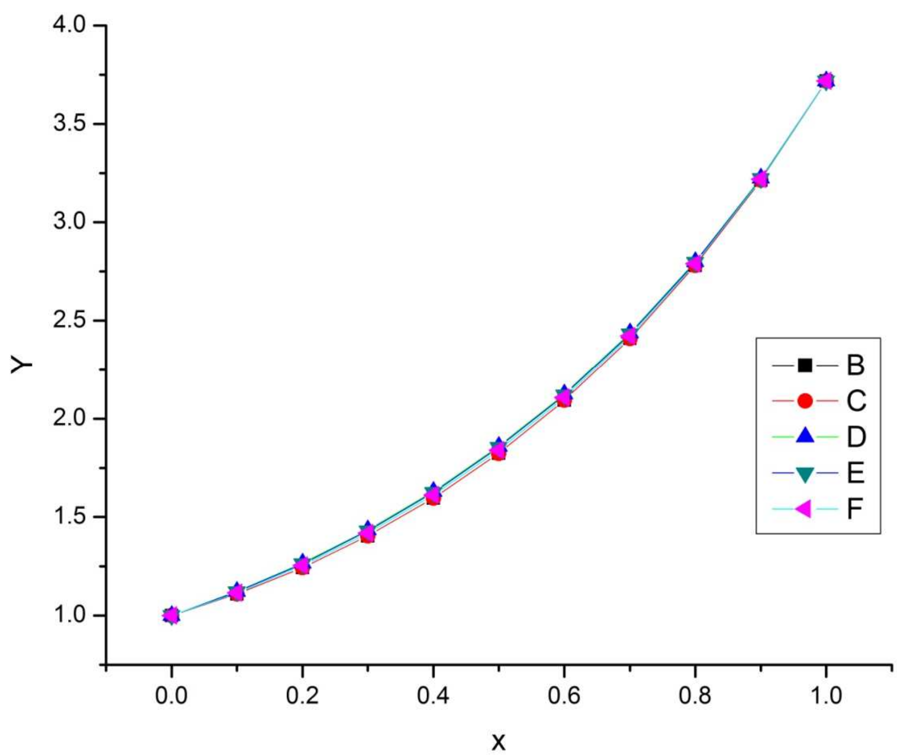
Table 2: Numerical results of Example 1 for different fractional orders

x	$y_{3.25}$	Error$y_{3.25}$	$y_{3.50}$	Error$y_{3.50}$	$y_{3.75}$	Error $y_{3.75}$
0.0	1.000000000	0.0000	1.0000000000	0.0000	1.0000000000	0.00000
0.1	1.1224993757	1.19E-2	1.1203115989	9.79E-3	1.1158792778	5.36E-3
0.2	1.2664588620	2.21E-2	1.2625329711	1.82E-2	1.2543233221	1.00E-2
0.3	1.4346750116	2.90E-2	1.4295456992	2.45E-2	1.4185517318	1.35E-2
0.4	1.6308722515	3.41E-2	1.6250890574	2.83E-2	1.6124751966	1.57E-2
0.5	1.8596278394	3.52E-2	1.8537329408	2.93E-2	1.8407263922	1.63E-2
0.6	2.1264040614	3.30E-2	2.1209141736	2.76E-2	2.1087179689	1.54E-2
0.7	2.4376149958	2.70E-2	2.4330014222	2.33E-2	2.4227179162	1.30E-2
0.8	2.8007157398	2.02E-2	2.7973805229	1.69E-2	2.7899399902	9.50E-3
0.9	3.2243108901	1.06E-2	3.2225578949	8.91E-3	3.2186490198	5.00E-3
1.0	3.7182818285	0.0000	3.7182818285	0.0000	3.7182818285	0.00000

Table 1 shows the comparison of the absolute error between exact solution and approximate solution for $\alpha = 4$. Here we used $M = 19$ and $k = 1$. Here y_{exact} and y_4 represent the exact solution and approximate solution of the problem at $\alpha = 4$. The numerical results given by the present method are also compared with MOHPM solutions. From the table it is obvious that the results of the current method are far better than MOHPM method.

In **Table 2 we** display the approximate solutions $y_{3.25}, y_{3.50}$ and $y_{3.75}$ for different values of $\alpha = 3.25, 3.50$ and 3.75 respectively. The errors, Error$y_{3.25,}$ Error$y_{3.50,}$ and Error$y_{3.75,}$ are the errors obtained for different values of $\alpha = 3.25, 3.50$ and 3.75 respectively. These errors are computed by comparing the approximate solutions obtained by the present method with the exact solutions of the given problem.

Fig.1. Solution graph to Example 1 for different order α.



In figure 1, the solution graph for different fractional order is presented. Here B shows the solution graph of exact solution. C, D, E and F are the approximate solution graphs of fractional orders differential equations having orders $\alpha = 4, 3.25, 3.50$ and 3.75 respectively.

Example 2 Consider the following fractional order boundary value problem [26].

$$\frac{d^{\alpha}y}{dx^{\alpha}} - 1 - \int_0^x e^{-t}u(t)dt = 0, \qquad 3 < \alpha \leq 4,\ 0 \leq x \leq 1,$$

subject to the boundary conditions

$$y(0) = 1, y(1) = e, y''(0) = 1, y''(1) = e.$$

The analytic solution to this problem is given by

$$y(x) = e^{x}.$$

Table 3: Numerical results of Example 1for $\alpha = 4$

x	y_{exact}	y_4	Error y_4	Error MOHPM
0.0	1.000000000000000	1.000000000000000	0.000000	0.00000
0.1	1.105170918075648	1.105170918075110	5.37E-13	-3.3E-12
0.2	1.221402758160170	1.221402758159249	9.20E-13	-1.4E-11
0.3	1.349858807576003	1.349858807574855	1.14E-12	-3.0E-11
0.4	1.491824697641270	1.491824697640031	1.23E-12	-4.4E-11
0.5	1.648721270700128	1.648721270698916	1.21E-12	-4.3E-11
0.6	1.822118800390509	1.822118800389421	1.08E-12	-2.4E-11
0.7	2.013752707470477	2.013752707469592	8.84E-13	6.4E-12
0.8	2.225540928492468	2.225540928491845	6.22E-13	3.1E-11
0.9	2.459603111156950	2.459603111156629	3.20E-13	3.1E-11
1.0	2.718281828459045	2.718281828459045	1.00E-19	-1.6E-13

Table 3, illustrates the comparison of the absolute error between the approximate solution obtained by the current method and exact solution of the problem. The results reveal that the present method has higher degree of accuracy. In this example, we used M=19 and k=1, to obtain the approximations in table 3. y_{exact} and y_4 are the exact and approximate solution of example 2 respectively at $\alpha = 4$.

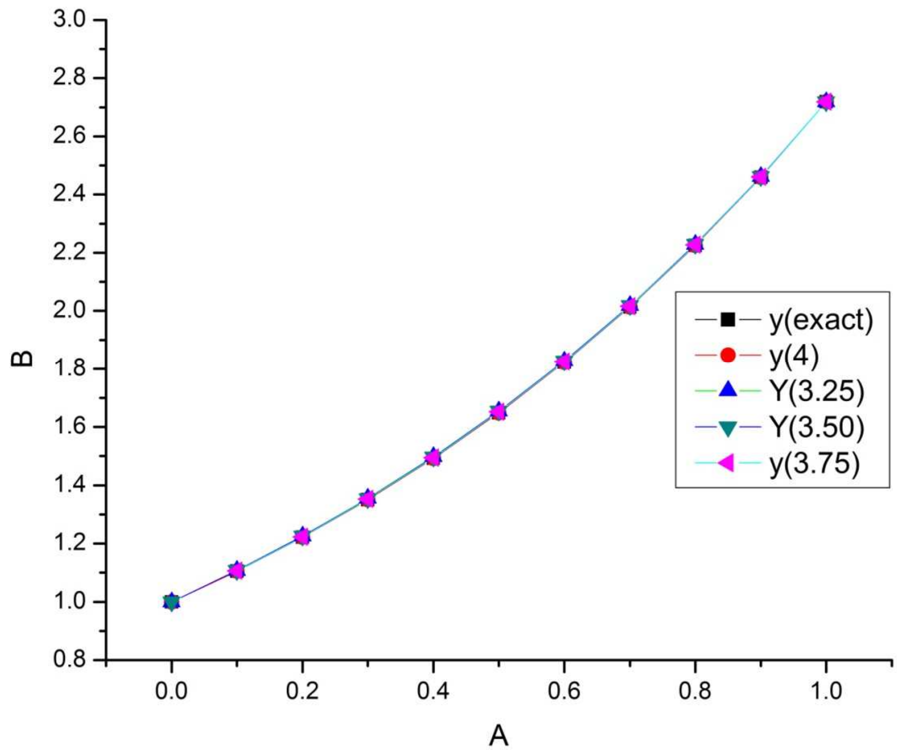
Error y_4 and Error MOHPM are the errors, obtained by the present method and the MOHPM method respectively. The table also show that the present method have greater accuracy than MOHPM method.

Table 4: Numerical results of Example 2 for different fractional order α.

x	$y_{3.25}$	Error$y_{3.25}$	$y_{3.50}$	Error$y_{3.50}$	$y_{3.75}$	Error $y_{3.75}$
0.0	0.9999999999	6.00E-20	1.0000000000	1.00E-19	0.00000000	0.00000
0.1	1.1073226891	2.15E-3	1.1070334032	1.86E-3	1.1062221795	1.05E-3
0.2	1.2253179326	3.91E-3	1.2248445375	7.91E-3	1.2233617284	1.95E-3
0.3	1.3549978427	5.13E-3	1.3544457773	1.09E-3	1.3524926035	2.63E-3
0.4	1.4976055951	5.78E-3	1.4970545308	4.32E-3	1.4948522387	3.02E-3
0.5	1.4976055951	5.85E-3	1.6540758406	1.22E-3	1.6518435878	3.12E-3
0.6	1.8275123942	5.39E-3	1.8271030150	3.19E-3	1.8250433402	2.92E-3
0.7	2.0182332007	4.48E-3	2.0179255501	4.40E-3	2.0162138012	2.46E-3
0.8	2.2287433691	3,20E-3	2.2285412843	3.88E-3	2.2273175384	1.77E-3
0.9	2.4612709967	1.66E-3	2.4611719862	1.71E-3	2.4605345410	9.31E-4
1.0	2.7182818284	2.00E-19	2.7182818284	2.0E-19	2.7182818284	4.0E-19

Table 4 represent the solution of different fractional order differential equations of different orders, $\alpha = 3.25, 3.50$ and 3.75. $y_{3.25}$, $y_{3.50}$, $y_{3.75}$ are the solutions and Error $y_{3.25}$, Error $y_{3.50}$ and Error $y_{3.75}$ are the corresponding errors of the fractional order differential equations at $\alpha = 3.25, 3.50, 3.75$ respectively.

Fig. 2: The solution graph of Example 2, for different order α and exact solution.



Example 3

Consider the fractional order nonlinear volterra-Fredholm integro-differential equation [27]:

$$x^4 \frac{d^\alpha}{dx^\alpha} y - \frac{d^2}{dx^2} y + \frac{dy}{dx} + \frac{x^6}{30} + \frac{x^4}{6} + \frac{x^2}{2} - \frac{14x}{3} + \frac{1}{2} - \int_0^x (x-t)\big(y(t)\big)^2 dt + 2\int_0^x (x+t)y(t)dt, \qquad 3 < \alpha \le 4, \qquad 0 \le x \le 1.$$

subject to the initial conditions

$$y(0) = 1, y'(0) = 0, y''(0) = 2\,, y'''(0) = 0,$$

The exact solution is

$$y(x) = 1 + x^2.$$

Table 5: The Numerical solutions of Example 3 for $\alpha = 4$

x	y_{exact}	y_4	Error y_4	
0.0	0.9999999999999999999	1.000000000000000000	6.00E-20	
0.1	1.0099999999999999996	1.010000000000000000	4.00E-19	
0.2	1.0399999999999999973	1.040000000000000000	2.70E-18	
0.3	1.0899999999999999814	1.090000000000000000	1.86E-17	
0.4	1.1599999999999998911	1.160000000000000000	1.08E-16	
0.5	1.2499999999999995919	1.250000000000000000	4.08E-16	
0.6	1.3599999999999988776	1.360000000000000000	1.12E-15	
0.7	1.4899999999999974841	1.490000000000000000	2.51E-15	
0.8	1.6399999999999951020	1.640000000000000000	4.89E-15	
0.9	1.8099999999999913883	1.810000000000000000	8.61E-15	
1.0	1.9999999999999859746	2.000000000000000000	1.40E-14	

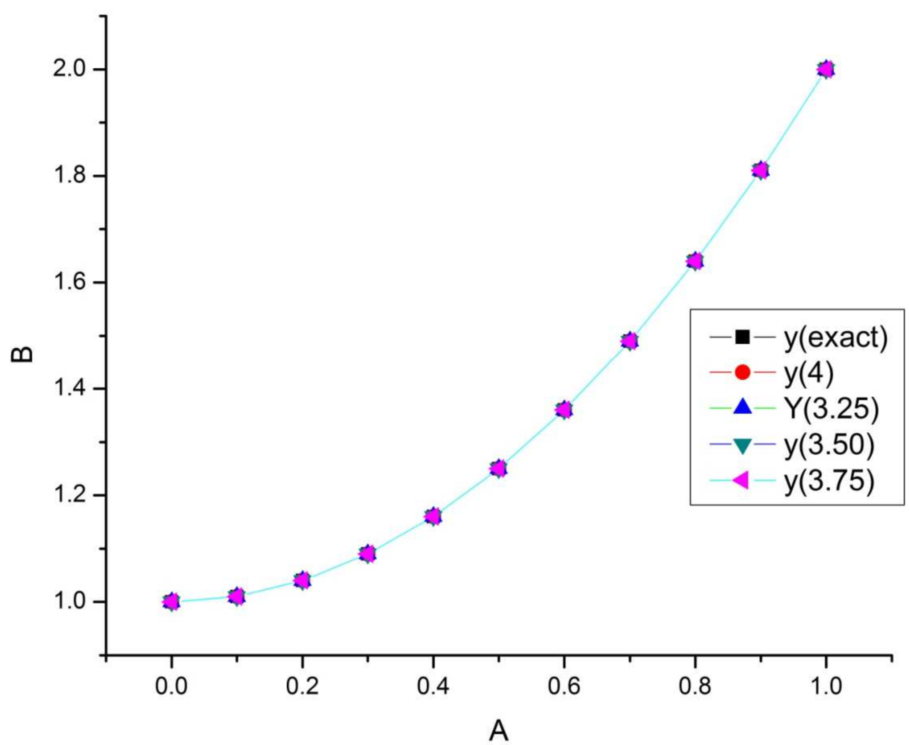
In Table 5, we have presented y_{exact} the exact solution and y_5 the approximate solution for example 3. Here we use $M = 19$ and $k = 1$ for the implementation 0f the current method. The approximate solution is compared with the exact solution. This table reveal that the present method have the highest degree of accuracy. The exact and approximate solutions at $\alpha = 4$ are represented by y_{exact} and y_4 respectively. Error y_4 is the error associated with the present method.

Table 6: Numerical results of Example 3, for different fractional orders

x	$y_{3.25}$	Error$y_{3.25}$	$y_{3.50}$	Error$y_{3.50}$	$y_{3.75}$	Error $y_{3.75}$
0.0	1.00000000000000000020	2.00E-19	1.00000000000000	0.0000	1.0000000000000	2.0E-19
0.1	1.00999999999999998970	1.03E-17	1.01000000000000	1.7E-17	1.0099999999999	4.1E-18
0.2	1.03999999999999991310	8.69E-17	1.04000000000000	1.5E-16	1.0399999999999	4.2E-17
0.3	1.08999999999999973210	2.67E-16	1.09000000000000	5.2E-16	1.0899999999999	1.4E-16
0.4	1.15999999999999942530	5.74E-16	1.16000000000000	1.1E-15	1.1599999999999	3.5E-16
0.5	1.24999999999999897420	1.02E-15	1.25000000000000	2.2E-15	1.2499999999999	6.8E-16
0.6	1.35999999999999836190	1.63E-15	1.36000000000000	3.6E-15	1.3599999999999	1.1E-15
0.7	1.48999999999999757190	2.42E-15	1.49000000000000	5.6E-15	1.4899999999999	1.8E-15
0.8	1.63999999999999658810	3.41E-15	1.64000000000000	8.1E-15	1.6399999999999	2.7E-15
0.9	1.80999999999999539480	4.60E-15	1.81000000000001	1.1E-15	1.8099999999999	3.8E-15
1.0	1.99999999999999397690	6.02E-15	2.00000000000001	1.5E-14	1.9999999999999	5.2E-15

Table 6 represents the solution of fractional order differential equations for different values of $\alpha = 3.25, 3.50$ and 3.75. $y_{3.25}$, $y_{3.50}$, $y_{3.75}$ and the errors, Error $y_{3.25}$, Error $y_{3.50}$, and Error $y_{3.75}$ are the solutions and errors at $\alpha = 3.25, 3.50, 3.75$ respectively.

Fig. 3: The solution graph of Example 3.



Example 4

Consider the fractional order nonlinear Volterra-Fredholm integro-differential equation [27]

$$\frac{d^{\alpha}}{dx^{\alpha}}y+y+\frac{x^5}{5}-\frac{2x^3}{3}-\frac{5x^2}{6}+\frac{113x}{105}+1-\int_0^x y^2(t)dt-\int_0^1 xt(x+t)y^2(t)dt=0,$$

$$2<\alpha\le 3, 0\le x\le 1,$$

subject to the initial conditions

$$y(0)=-1, y'(0)=0, y''(0)=2\,,$$

the exact solution is

$$y(x)=-1+x^2.$$

Table 7: Numerical results of Example 4, for $\alpha=3$.

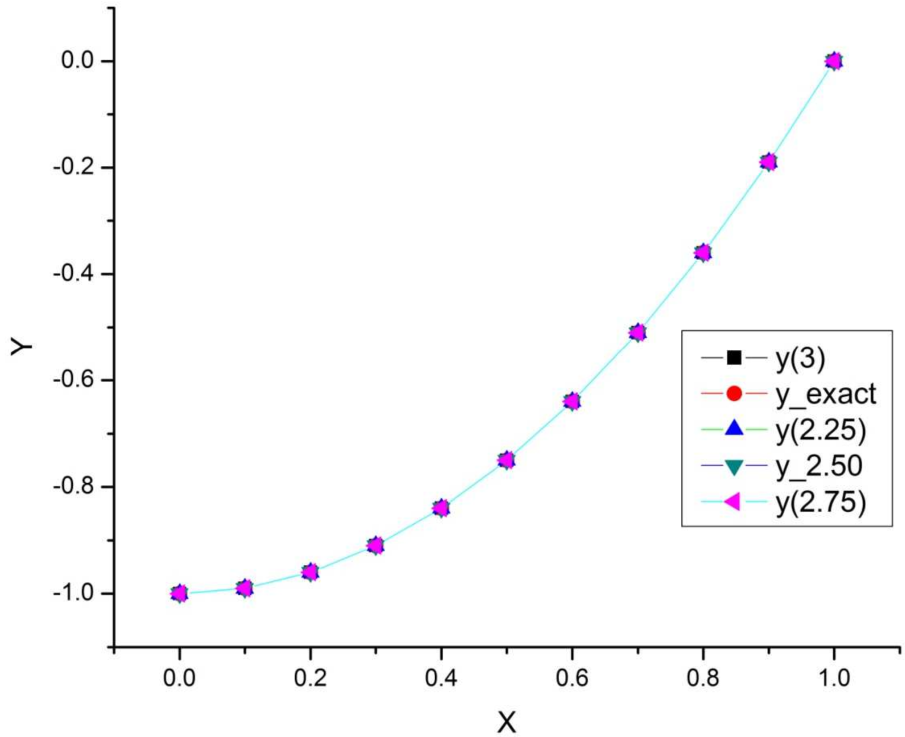
x	y_3	y_{exact}	Error y_3
0.0	-0.9999999999999999999	-1.0000000000000000000	8.00E-20
0.1	-0.9900000000000000690	-0.9900000000000000000	6.90E-17
0.2	-0.9600000000000003510	-0.9600000000000000000	3.51E-16
0.3	-0.9100000000000008588	-0.9100000000000000000	8.58E-16
0.4	-0.8400000000000015919	-0.8400000000000000000	1.59E-15
0.5	-0.7500000000000025491	-0.7500000000000000000	2.54E-15
0.6	-0.6400000000000037283	-0.6400000000000000000	3.72E-15
0.7	-0.5100000000000051264	-0.5100000000000000000	5.12E-15
0.8	-0.3600000000000067388	-0.3600000000000000000	6.73E-15
0.9	-0.1900000000000085599	-0.1900000000000000000	8.55E-15
1.0	-0.0000000000000105818	0.0000000000000000000	1.05E-14

In Table 7, we have listed the values of y_{exact} the exact solution and y_3 the approximate solution for example 4. Here we use $M=19$ and $k=1$ for the implementation of the current method. The accuracy is compared with the exact solutions of the problem. From the table it is cleared that the present method has the highest degree of accuracy. The exact and approximate solutions at $\alpha=3$ are represented by y_{exact} and y_3 respectively. The error associated with the present method is represented by Errory_3.

Table 8: Numerical results for Example 4 for different fractional orders.

x	$y_{2.25}$	Error$y_{2.25}$	$y_{2.50}$	Error$y_{2.50}$	$y_{2.75}$	Error $y_{2.75}$
0.0	-1.00000000000000000000	0.00000	-1.00000000000000	0.0000	-0.9999999999999	1.5E-19
0.1	-0.99000000000000013351	1.3E-16	-0.99000000000000	4.7E-17	-0.9899999999999	3.9E-17
0.2	-0.95999999999999999844	1.5E-18	-0.95999999999999	5.9E-19	-0.9600000000000	1.2E-19
0.3	-0.90999999999999999997	3.0E-20	-0.91000000000000	1.7E-19	-0.9099999999999	5.1E-20
0.4	-0.84000000000000000001	1.0E-20	-0.84000000000000	1.4E-19	-0.8399999999999	1.0E-19
0.5	-0.74999999999999999998	2.0E-20	-0.75000000000000	6.1E-20	-0.7499999999999	1.0E-19
0.6	-0.64000000000000000000	0.00000	-0.63999999999999	2.1E-20	-0.6399999999999	2.3E-19
0.7	-0.51000000000000000000	0.00000	-0.51000000000000	1.3E-19	-0.5099999999999	1.8E-19
0.8	-0.35999999999999999965	3.5E-19	-0.35999999999999	6.1E-20	-0.3600000000000	2.3E-19
0.9	-0.18999999999999999244	7.5E-18	-0.18999999999999	3.4E-18	-0.1900000000000	7.5E-18
1.0	0.00000000000000210330	2.1E-15	0.00000000000000	9.2E-16	-0.0000000000000	2.0E-15

Table 8 represents the solution of fractional order differential equations for different values of $\alpha=2.25, 2.50$ and 2.75. The solutions, $y_{2.25}$, $y_{2.50}$ and $y_{2.75}$ and errors, Error $y_{2.25}$, Error $y_{2.50}$ and Error $y_{2.75}$ are the corresponding errors of fractional order differential equations at $\alpha=2.25, 2.50$ and 2.75 respectively.



Conclusion

In this research paper, we attempted to find the numerical solutions of fractional integro differential equations by using Chebyshev Wavelet Method (CWM). The Caputo differential operator is used to define fractional derivative in each fractional integro differential equation. The simulations performed by Chebyshev Wavelet method are simple, straight forward and effective. The numerical results obtained by current method are compared with the results of MOHPM. The comparison shows that the present method has higher degree of accuracy than MOHPM. The absolute error is calculated, which confirmed that the solution obtained by the propose method are strongly agree with the exact solution of the problems.

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