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Numerical Treatment of an Epidemic Model with Spatial Diffusion

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ABSTRACT

Structure preserving numerical methods are of great concern now a day. Many physical systems possess different properties which must be preserved by numerical method. For example, positivity is an important physical property possessed by different models. For instance, negative values for concentration of chemical reactions and subpopulations of an epidemic model cannot be negative. The purpose of this work is to propose a structure preserving numerical scheme for the solution of a reaction-diffusion epidemic model with specific nonlinear incidence rate. The proposed method is an implicit finite difference (FD) scheme. The proposed scheme is unconditionally dynamical consistent with positivity property. The proposed FD scheme is unconditionally convergent to the true steady states of the SIR reaction-diffusion system with specific nonlinear incidence rate. Comparison of proposed FD scheme is also done with the classical finite difference schemes to verify all the claims.

KEYWORDS: SIR epidemic model with specific nonlinear incidence rate; finite difference method; Convergence; Positivity.

1. INTRODUCTION

Mathematical epidemic models play a key role in the understanding of the dynamics of spread and control for the infectious diseases. Several mathematical epidemic models are presented in the literature to understand different infectious disease dynamics [3-7]. As nonlinearity on incidence rates of different diseases was examined, several writers have proposed the modification in standard bilinear incidence rate. In this work, we consider the following SIR reaction-diffusion epidemic model with specific nonlinear incidence rate for the numerical solution proposed by Al Mehdi Lotfi et al [2].

$$\left.\begin{aligned} \frac{\partial S}{\partial t} &= \lambda - \pi S - \frac{\beta SI}{1+\vartheta_1 S+\vartheta_2 I+\vartheta_3 SI} + \alpha_1 \frac{\partial^2 S}{\partial x^2} \\ \frac{\partial I}{\partial t} &= \frac{\beta SI}{1+\vartheta_1 S+\vartheta_2 I+\vartheta_3 SI} - (\pi+\delta+r)I + \alpha_2 \frac{\partial^2 I}{\partial x^2} \\ \frac{\partial R}{\partial t} &= rI - \pi R + \alpha_3 \frac{\partial^2 R}{\partial x^2} \end{aligned}\right\} \tag{1.1}$$

Where $S = S(x,t), I = I(x,t)$ and $R = R(x,t)$ are susceptible, infectious and recovered individuals respectively.λ is the recruitment rate, π is the natural death rate, δ is the death rate due to disease, r is the recovery rate of infected persons, β is infection parameter and $\beta SI/(1+\vartheta_1 S+\vartheta_2 I+\vartheta_3 SI)$ is the incidence rate, where $\vartheta_1, \vartheta_2, \vartheta_3 \geq 0$ are constants. Remember that the above incidence rate becomes bilinear if $\vartheta_i = 0, i = 1,2,3$ and the saturated incidence if $\vartheta_i = 0, i = 1,2$ or $i = 2,3$ [2].

In terms of numerical solutions for homogeneous and nonhomogeneous epidemic model, different authors proposed different numerical techniques [8-11]. But in this work our focus is to discuss a positivity preserving numerical techniques and propose an efficient positivity preserving numerical scheme for the solution of the reaction-diffusion epidemic system (1.1). Several authors presented different positivity preserving explicit and implicit numerical techniques for the ordinary and partial differential equations [13-17,19-24]. Mickens [12] presented rules to construct positivity persevering finite difference schemes, called nonstandard finite difference (NSFD) schemes. Mainly, he suggested that nonlinear term should replace with nonlocal approximation and discrete representation of derivative have non-trivial denominator functions. NSFD schemes are also applied on different epidemic models by several authors (for the readers, reference thereon [1,14-16,19]).

In this paper, a novel and efficient positivity preserving FD scheme is used to solve the reaction-diffusion epidemic system (1.1) which is proposed by Settapat Chinviriyasit and Wirawanchinviriyasit [17]. The proposed FD scheme is unconditionally stable, unconditionally positivity preserving and unconditionally convergent to

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the true steady states of the continuous system (1.1). Since R is not present in first two equations, so we can rewrite system (1.1) as

$$\frac{\partial S}{\partial t} = \lambda - \pi S - \frac{\beta SI}{1+\vartheta_1 S+\vartheta_2 I+\vartheta_3 SI} + \alpha_1 \frac{\partial^2 S}{\partial x^2}, \quad 0 < x < L \tag{1.2}$$

$$\frac{\partial I}{\partial t} = \frac{\beta SI}{1+\vartheta_1 S+\vartheta_2 I+\vartheta_3 SI} - (\pi + \delta + r)I + \alpha_2 \frac{\partial^2 I}{\partial x^2}, \quad 0 < x < L \tag{1.3}$$

With initial conditions

$$S(x,0) = \sigma_1(x) \text{ and } I(x,0) = \sigma_2(x) \tag{1.4}$$

And homogenous boundary conditions

$$S_x(0,t) = S_x(L,t) = 0, \tag{1.5}$$

$$I_x(0,t) = I_x(L,t) = 0, \tag{1.6}$$

2. Analysis of the model

The system (1.2) - (1.3) has two equilibrium points, disease free equilibrium (DFE) point and endemic equilibrium (EE) point [2]. The DFE point for the system (1.2) – (1.3) is

$E_{DFE}(S_{DFE}, I_{DFE}) = E_{DFE}(\lambda/\pi, 0)$ and EE point is $E_{EE}(S_{EE}, I_{EE})$, where

$$S_{EE} = \frac{2(e+\vartheta_2\lambda)}{\beta-\vartheta_1 e+\vartheta_2\pi-\vartheta_3\lambda+\sqrt{d}}$$

And

$$I_{EE} = \frac{\lambda-\pi S_{EE}}{e}$$

With $e = (\pi + \delta + r)$ and $d = (\beta - \vartheta_1 e + \vartheta_2\pi - \vartheta_3\lambda)^2 + 4\vartheta_3\pi(e + \vartheta_2\lambda)$

Here, the reproductive number of the system [2] is

$$\Re_0 = \frac{\beta\lambda}{(\pi + \vartheta_1\lambda)(\pi + \delta + r)}$$

3. Numerical Modeling

In this section, three finite difference (FD) methods are used to solve the system (1.2) – (1.3), i.e backward Euler FD method, Crank Nicolson FD method and proposed positivity preserving FD method. We propose positivity preserving FD method for the numerical solution of the system (1.2) – (1.3) and then we will compare our results with a standard backward Euler and Crank Nicolson FD methods in this paper.

3.1. Backward Euler FD Method

Divide $[0,L] \times [0,T]$ into $M \times N$ with step sizes $h = \frac{L}{M}$ and $\tau = \frac{T}{N}$.

Grid points are

$x_i = ih, \quad i = 0,1,2,\dots,M,$

$t_n = n\tau, \quad n = 0,1,2,\dots,N,$

S_i^n and I_i^n are denoted as FD approximations of $S(ih, n\tau)$ and $I(ih, n\tau)$ respectively.

In the current section, we present backward Euler FD method for reaction-diffusion epidemic system (1.2) – (1.3) which is given as

$$(1 + 2\eta_1)S_i^{n+1} - \eta_1(S_{i-1}^{n+1} + S_{i+1}^{n+1}) = S_i^n + \lambda\tau - \tau\pi S_i^n - \frac{\beta\tau S_i^n I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n}$$

$$(1 + 2\eta_2)I_i^{n+1} - \eta_2(I_{i-1}^{n+1} + I_{i+1}^{n+1}) = I_i^n - \tau(\pi + \delta + r)I_i^n + \frac{\beta\tau S_i^n I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n}$$

3.2. Crank Nicolson FD Method

In this section, the Crank Nicolson FD scheme for the reaction-diffusion system (1.2) – (1.3) is given as

$$(1 + \eta_1)S_i^{n+1} - \frac{\eta_1}{2}(S_{i-1}^{n+1} + S_{i+1}^{n+1}) = (1 - \eta_1)S_i^n + \frac{\eta_1}{2}(S_{i-1}^n + S_{i+1}^n) + \lambda\tau - \tau\pi S_i^n - \frac{\beta\tau S_i^n I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n}$$

$$(1 + \eta_2)I_i^{n+1} - \frac{\eta_2}{2}(I_{i-1}^{n+1} + I_{i+1}^{n+1}) = (1 - \eta_2)I_i^n + \frac{\eta_2}{2}(I_{i-1}^n + I_{i+1}^n) - \tau(\pi + \delta + r)I_i^n + \frac{\beta\tau S_i^n I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n}$$

3.3. Proposed FD Method

Now we present proposed FD scheme for SIR reaction-diffusion epidemic system (1.2) – (1.3) with specific nonlinear incidence rate. The proposed FD scheme is implicit FD scheme which is given as

$$-\eta_1 S_{i-1}^{n+1} + (1 + 2\eta_1)S_i^{n+1} - \eta_1 S_{i+1}^{n+1} = S_i^n + \lambda\tau - \tau\pi S_i^{n+1} - \frac{\beta\tau S_i^{n+1} I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n} \tag{3.3.1}$$

$$-\eta_2 I_{i-1}^{n+1} + (1 + 2\eta_2)I_i^{n+1} - \eta_2 I_{i+1}^{n+1} = I_i^n - \tau(\pi + \delta + r)I_i^{n+1} + \frac{\beta\tau S_i^n I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n} \tag{3.3.2}$$

M-matrix theory [18] can be helpful to prove the positivity of the discretized system (3.3.1) – (3.3.2). A square matrix having real entries is call M-matrix if entries in off-diagonal are non-positive, entries in diagonal are positive and matrix is strictly diagonally dominant. If a matrix is M-matrix, then it is singular. Inverse matrix of M-matrix always has the positive numbers entries.

Theorem 3.3.1 [19]
For any $h > 0$ and $\tau > 0$, the system (3.4.1) – (3.4.2) is positive, i.e. $S^n > 0$ and $I^n > 0$ for all $n = 0,1,...$
Proof
The system (3.3.1) – (3.3.2) can be written as
$AS^{n+1} = B$and (3.3.3)
$CI^{n+1} = D$ (3.3.4)
Here A and B are square matrices of dimension $(N+1) \times (N+1)$. B and D are column matrices.

$$A = \begin{pmatrix} a_3 & a_1 & 0 & & 0 & 0 & 0 \\ a_2 & a_3 & a_2 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & a_3 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & a_3 & a_2 & 0 \\ 0 & 0 & 0 & \cdots & a_2 & a_3 & a_2 \\ 0 & 0 & 0 & & 0 & a_1 & a_3 \end{pmatrix}$$

$$C = \begin{pmatrix} c_3 & c_1 & 0 & & 0 & 0 & 0 \\ c_2 & c_3 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & c_3 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & c_3 & c_2 & 0 \\ 0 & 0 & 0 & \cdots & c_2 & c_3 & c_2 \\ 0 & 0 & 0 & & 0 & c_1 & c_3 \end{pmatrix}$$

The off-diagonal entries and diagonal entries of A are $a_1 = -2\eta_1$, $a_2 = -\eta_1$ and $a_3 = 1 + 2\eta_1 + \tau\pi + \beta\tau I_i^n/(1 + \vartheta_1 S_i^n + \vartheta_2 I_i^n \vartheta_3 S_i^n I_i^n)$. The off-diagonal entries and diagonal entries of C are $c_1 = -2\eta_2$, $c_2 = -\eta_1$ and $c_3 = 1 + 2\eta_1 + \tau(\pi + \delta + r)$. Where $\eta_1 = \alpha_1\tau/h^2$ and $\eta_2 = \alpha_2\tau/h^2$. The entries of column matrix B are $S_i^n + \lambda\tau$ and entries of column matrix D are $I_i^n + \frac{\beta\tau S_i^n I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n}$. Also $S^n = (S_0^n, S_1^n, \dots, S_M^n)^T$ and $I^n = (I_0^n, I_1^n, \dots, I_M^n)^T$. Since $S_i^0 \geq 0$ and $I_i^0 \geq 0$ so $a_3 > 0$ and obviously $c_3 > 0$. Also $a_1, a_2, c_1, c_2 < 0$ and A, B are strictly diagonally dominant. From all the above conditions, it is concluded that A and C are M-matrices. This implies that A and C are non-singular matrices.So the equations (3.3.3) and (3.3.4) can be written as
$S^{n+1} = A^{-1}B$ (3.4.5)
$I^{n+1} = C^{-1}D$ (3.4.6)

Suppose that $S^n > 0$ and $I^n > 0$ and since A and C are M-matrix therefore all the entries of A^{-1} and C^{-1} are positive. So it is concluded that $S^{n+1} > 0$ and $I^{n+1} > 0$. Thus by induction, the system is positive.

3.4. Order and Consistency of Proposed FD Method

Accuracy of the proposed FD scheme can be obtained with the supposition of local truncation error as

$$\mathcal{L}_S[S, I: h, \tau] = \frac{1}{\tau}[S_i^{n+1} - S_i^n] - \lambda\tau + \tau\pi S_i^{n+1} + \frac{\beta\tau S_i^{n+1} I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n} - \frac{\alpha_1}{h^2}[S_{i-1}^n - 2S_i^n + S_{i+1}^n] - \left[\frac{\partial S}{\partial t} - \lambda + \pi S + \frac{\beta SI}{1+\vartheta_1 S+\vartheta_2 I+\vartheta_3 SI} - \alpha_1 \frac{\partial^2 S}{\partial x^2}\right] \quad (3.4.1)$$

$$\mathcal{L}_I[S, I: h, \tau] = \frac{1}{\tau}[I_i^{n+1} - I_i^n] + \tau(\pi + \delta + r)I_i^{n+1} - \frac{\beta\tau S_i^n I_i^n}{1+\vartheta_1 S_i^n+\vartheta_2 I_i^n+\vartheta_3 S_i^n I_i^n} - \frac{\alpha_2}{h^2}[I_{i-1}^n - 2I_i^n + I_{i+1}^n] - \left[\frac{\partial I}{\partial t} - \frac{\beta SI}{1+\vartheta_1 S+\vartheta_2 I+\vartheta_3 SI} + (\pi + \delta + r)I - \alpha_2 \frac{\partial^2 I}{\partial x^2}\right]$$
(3.4.2)

The Taylor's series expansion of S_i^{n+1},S_{i-1}^{n+1},S_{i+1}^{n+1},I_i^{n+1},I_{i-1}^{n+1} and I_{i+1}^{n+1} are

$$S_i^{n+1} = S_i^n + \tau\frac{\partial S_i^n}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 S_i^n}{\partial t^2} + \frac{\tau^3}{2!}\frac{\partial^3 S_i^n}{\partial t^3} + \cdots,$$
$$S_{i+1}^{n+1} = S_i^n + \tau\frac{\partial S_i^n}{\partial t} + h\frac{\partial S_i^n}{\partial x} + \frac{\tau^2}{2!}\frac{\partial^2 S_i^n}{\partial t^2} + \frac{h^2}{2!}\frac{\partial^2 S_i^n}{\partial x^2} + \tau h\frac{\partial^2 S_i^n}{\partial x \partial t} + \cdots,$$
$$S_{i-1}^{n+1} = S_i^n + \tau\frac{\partial S_i^n}{\partial t} - h\frac{\partial S_i^n}{\partial x} + \frac{\tau^2}{2!}\frac{\partial^2 S_i^n}{\partial t^2} + \frac{h^2}{2!}\frac{\partial^2 S_i^n}{\partial x^2} - \tau h\frac{\partial^2 S_i^n}{\partial x \partial t} + \cdots,$$
$$I_i^{n+1} = I_i^n + \tau\frac{\partial I_i^n}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 I_i^n}{\partial t^2} + \frac{\tau^3}{2!}\frac{\partial^3 I_i^n}{\partial t^3} + \cdots,$$
$$I_{i+1}^{n+1} = I_i^n + \tau\frac{\partial I_i^n}{\partial t} + h\frac{\partial I_i^n}{\partial x} + \frac{\tau^2}{2!}\frac{\partial^2 I_i^n}{\partial t^2} + \frac{h^2}{2!}\frac{\partial^2 I_i^n}{\partial x^2} + \tau h\frac{\partial^2 I_i^n}{\partial x \partial t} + \cdots,$$

$$I_{i-1}^{n+1} = I_i^n + \tau\frac{\partial I_i^n}{\partial t} - h\frac{\partial I_i^n}{\partial x} + \frac{\tau^2}{2!}\frac{\partial^2 I_i^n}{\partial t^2} + \frac{h^2}{2!}\frac{\partial^2 I_i^n}{\partial x^2} - \tau h\frac{\partial^2 I_i^n}{\partial x\partial t} + \cdots,$$

Substituting the values of S_i^{n+1},S_{i-1}^{n+1},S_{i+1}^{n+1},I_i^{n+1},I_{i-1}^{n+1} and I_{i+1}^{n+1} in (3.4.1) and (3.4.2) and after simplifications we have,

$$\mathcal{L}_S[S, I: h, \tau] = -\frac{1}{12}\alpha_1 h^2\frac{\partial^4 S}{\partial x^4} + \tau\left[\frac{1}{2}\frac{\partial^2 S}{\partial t^2} + \pi\frac{\partial S}{\partial t} + \frac{\beta I}{1+\vartheta_1 S+\vartheta_2 I+\vartheta_3 SI}\frac{\partial S}{\partial t} - \alpha_1\frac{\partial^3 S}{\partial x^2\partial t}\right] + \cdots, \quad (3.4.3)$$

$$\mathcal{L}_I[S, I: h, \tau] = -\frac{1}{12}\alpha_2 h^2\frac{\partial^4 I}{\partial x^4} + \tau\left[\frac{1}{2}\frac{\partial^2 S}{\partial t^2} + (\pi+\delta+r)\frac{\partial S}{\partial t} - \alpha_2\frac{\partial^3 S}{\partial x^2\partial t}\right] + \cdots \quad (3.4.4)$$

From equations (3.4.3) and (3.4.4), it is verified that proposed FD method is $O(h^2+\tau)$ as $h, \tau \to 0$.

3.6. Stability of Proposed FD Method

To prove that the proposed scheme is unconditionally stable, we use Von Neumann stability method. For this purpose, we substitute $\zeta(t+\Delta t)e^{i\propto x}$, $\zeta(t)e^{i\propto x}$, $\zeta(t+\Delta t)e^{i\propto(x-\Delta x)}$ and $\zeta(t+\Delta t)e^{i\propto(x+\Delta x)}$in S_i^{n+1},S_i^n,S_{i-1}^{n+1} andS_{i+1}^{n+1} in equation (3.3.1). After linearizing and simplifications, we have

$$\left|\frac{\zeta(t+\Delta t)}{\zeta(t)}\right| = \left|\frac{1}{1+4\eta_1\sin^2(\propto\Delta x/2)+\tau\pi+\tau\beta}\right| < 1 \quad (3.6.1)$$

In similar way, for equation (3.3.2)

$$\left|\frac{\zeta(t+\Delta t)}{\zeta(t)}\right| = \left|\frac{1+\tau\beta}{1+4\eta_2\sin^2(\propto\Delta x/2)+\tau(\pi+\delta+r)}\right| < 1 \quad (3.6.2)$$

From equations (3.6.1) and (3.6.2), it is proved that proposed scheme is unconditionally stable.
In similar way, it can be proved that Crank Nicolson FD scheme and backward Euler FD scheme are unconditionally stable.

Table 1 (Disease Free Equilibrium$R_0 < 1$)

Parameter	λ	π	δ	β	r	ϑ_1	q_2	ϑ_1	ϑ_2	ϑ_3	α_1	α_2
Value	0.5	0.1	0.1	0.2	0.5	0.1	0.2	0.1	0.02	0.03	0.1	0.5

Table 2 (Endemic Equilibrium$R_0 > 1$)

Parameter	λ	π	δ	β	r	ϑ_1	q_2	ϑ_1	ϑ_2	ϑ_3	α_1	α_2
Value	0.5	0.1	0.1	0.6	0.5	0.1	0.2	0.1	0.02	0.03	0.1	0.5

4. Numerical experiment:

Now by using the values of parameters given in the table 1 and table 2 [2]we execute numerical experiment for all finite difference schemes. For this purpose, we take system (1.1) – (1.2) with homogeneous boundary conditions and initial conditions [2]

$$S(x,0) = \begin{cases} 1.1x, & 0 \le x < 0.5 \\ 1.1(1-x), & 0.5 \le x < 1 \end{cases}$$

$$I(x,0) = \begin{cases} 0.5x, & 0 \le x < 0.5 \\ 0.5(1-x), & 0.5 \le x < 1 \end{cases}$$

4.1. Backward Euler FD Scheme:

First we present the graphs of backward Euler FD scheme for both disease free equilibrium and endemic equilibrium.

4.1.1. Disease Free Equilibrium:

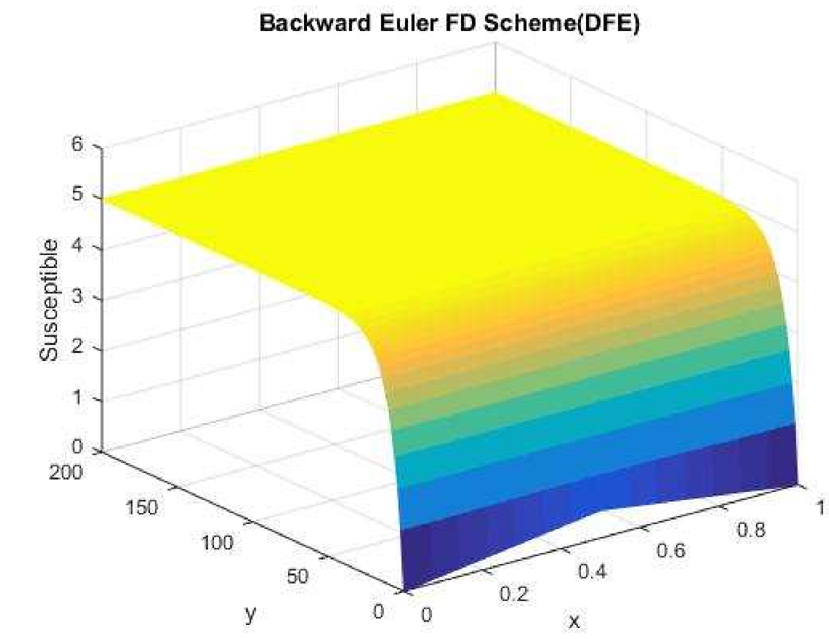

Figure 1(a)

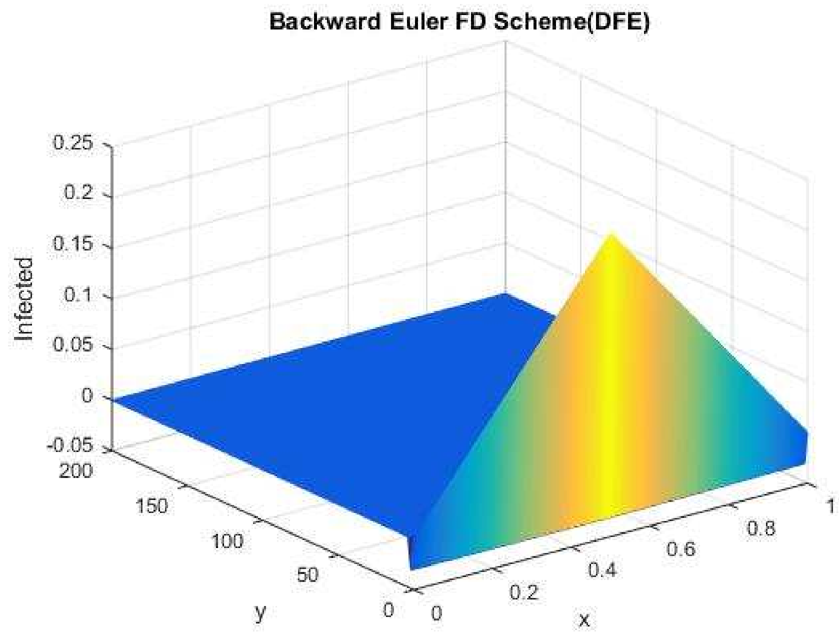

Figure 1(b)

Figure 1: **Figures 1(a) – 1(b) represent the graphs of susceptible and infected individuals for disease free equilibrium using backward Euler FD scheme at $h = 0.001$, $\eta_1 = 2 \times 10^{-5}$ and $\eta_2 = 10^{-6}$.**

Figure 1 shows that backward Euler FD scheme produces negative values of infected individuals which is meaningless in population dynamics. Therefore, backward Euler FD scheme fails to preserve positivity property possessed by system (1.2) – (1.3).

4.1.2. Endemic Equilibrium:

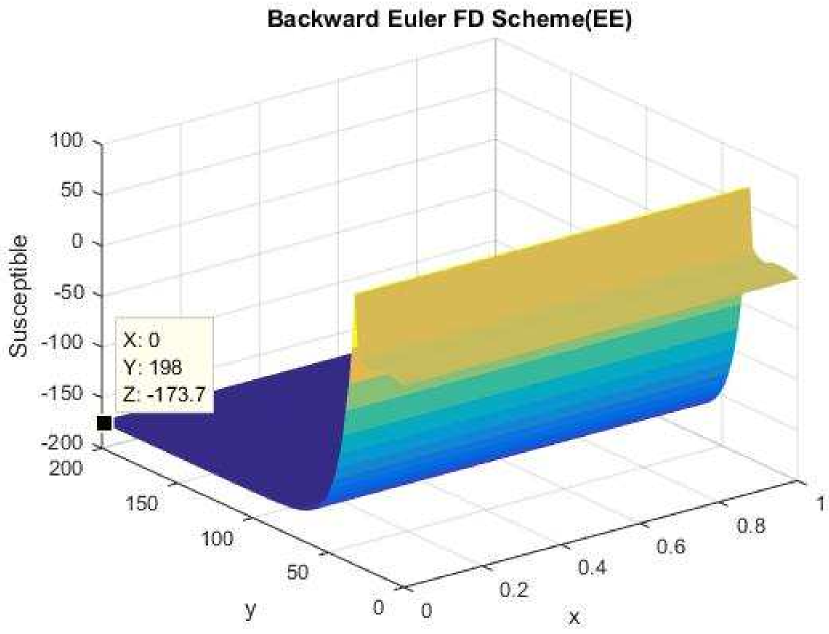

Figure 2(a)

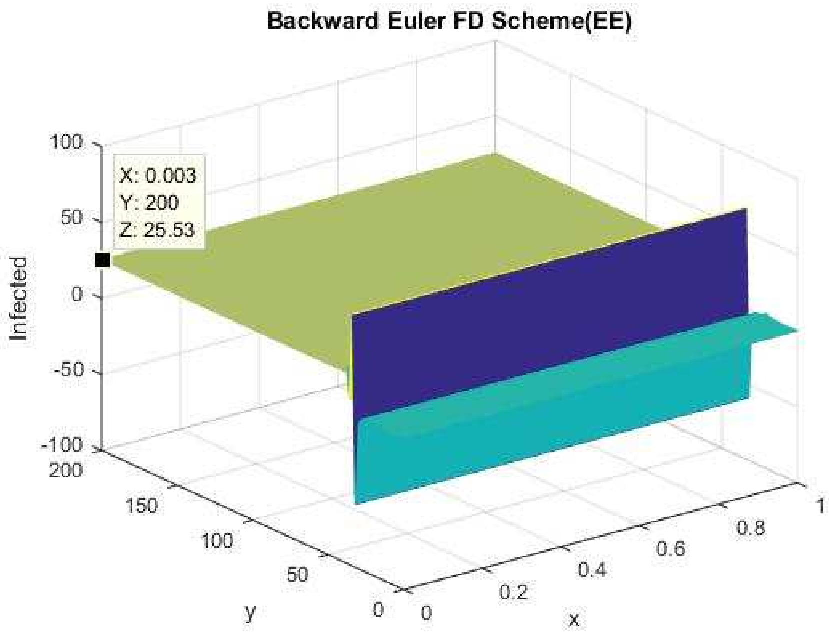

Figure 2(b)

Figure 2: Figures 2(a) – 2(b) show the graphs of susceptible and infected individuals for endemic equilibrium using backward Euler FD scheme at $h = 0.001$, $\eta_1 = 2 \times 10^{-5}$ and $\eta_2 = 10^{-6}$.

Figure 2 magnifies the graphs for endemic equilibrium of susceptible and infected individuals using backward Euler FD scheme. Figures 2(a) and 2(b) show that backward Euler FD scheme not only produces negative values of susceptible and infected individuals but also converges to false equilibrium point. Note that endemic equilibrium point is $E_{EE}(S_{EE}, I_{EE})$. After substituting the values of parameters in endemic point, we get $E_{EE}(S_{EE}, I_{EE}) = E_{EE}(1.3625, 0.5196)$. It can be verified from the figure 2 that backward Euler FD scheme converges to false endemic equilibrium point.

4.2. Crank Nicolson FD Scheme:

Now we present the graphs of Crank Nicolson FD scheme for both disease free equilibrium and endemic equilibrium.

4.2.1. Disease Free Equilibrium:

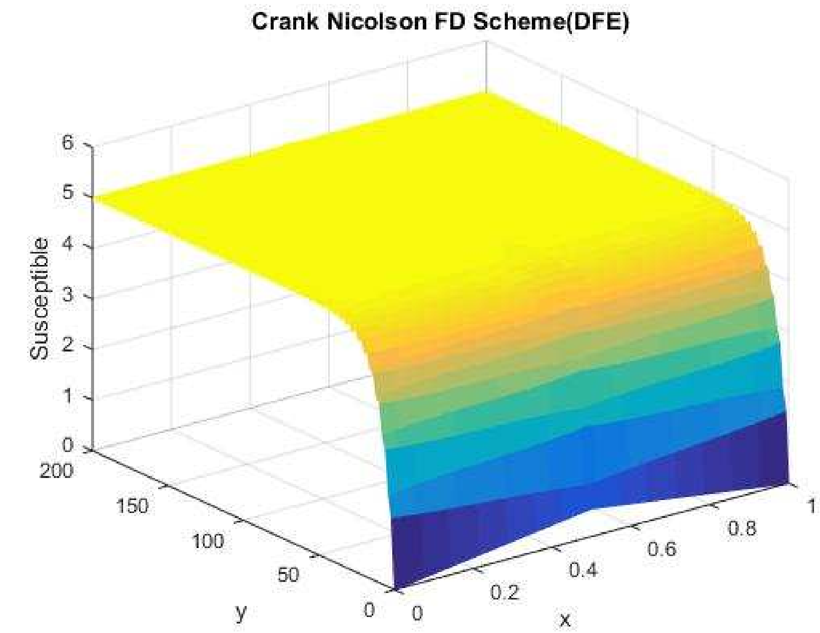

Figure 3(a)

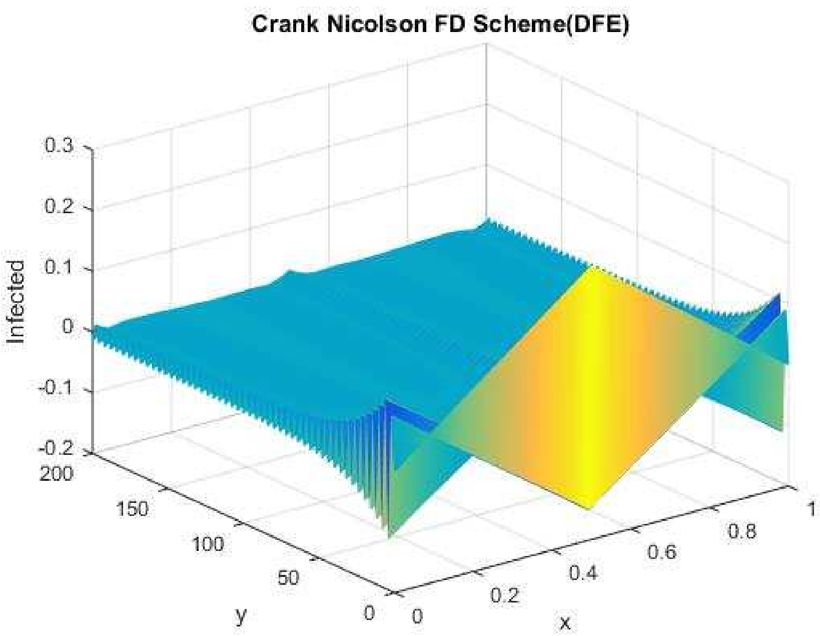

Figure 3(b)

Figure 3: Figures 3(a) – 3(b) represent the graphs of susceptible and infected individuals for disease free equilibrium using Crank Nicolson FD scheme at $h = 0.001$, $\eta_1 = 2 \times 10^{-5}$ and $\eta_2 = 10^{-6}$.

Crank Nicolson FD scheme also loses the positivity property as shown in the graph of infected individuals in figure 3(b).

4.2.2. Endemic Equilibrium:

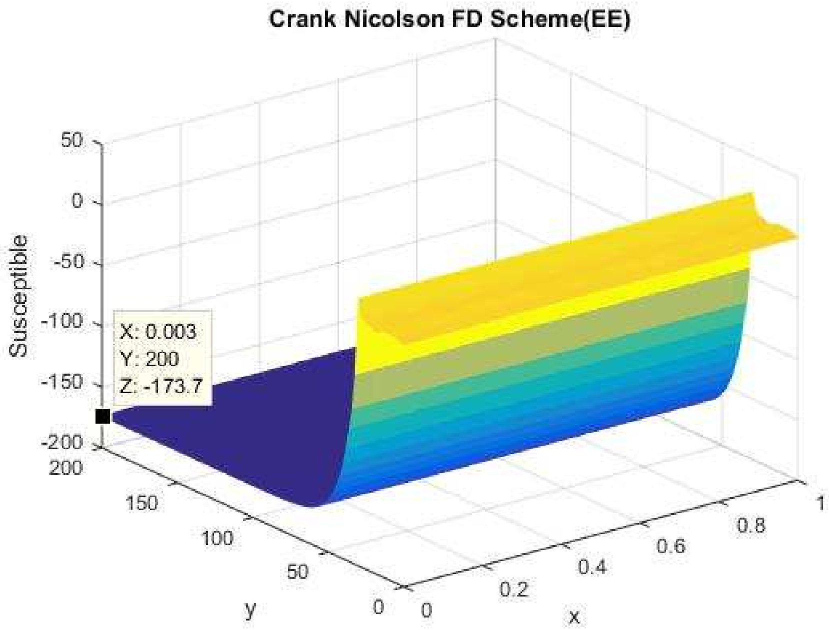

Figure 4(a)

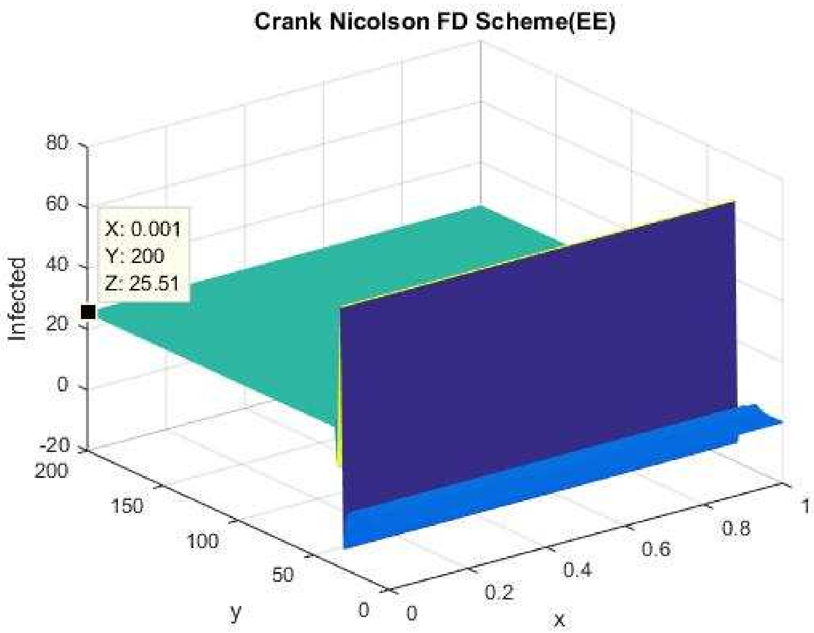

Figure 4(b)

Figure 4: Figures 4(a) – 4(b) represent the graphs of susceptible and infected individuals for endemic equilibrium using Crank Nicolson FD scheme at $h = 0.001$, $\eta_1 = 2 \times 10^{-5}$ and $\eta_2 = 10^{-6}$.

The graphs of susceptible and infected individuals in figure 4(a) and 4(b) indicate the failure of structure preserving properties by Crank Nicolson FD scheme, as Crank Nicolson FD scheme shows the negative behavior and converges to false equilibrium point.

4.3. Proposed FD Scheme:

Now we discuss the behavior of proposed FD scheme and present the simulations. First we present the graphs of initial distributions for susceptible and infected individuals.

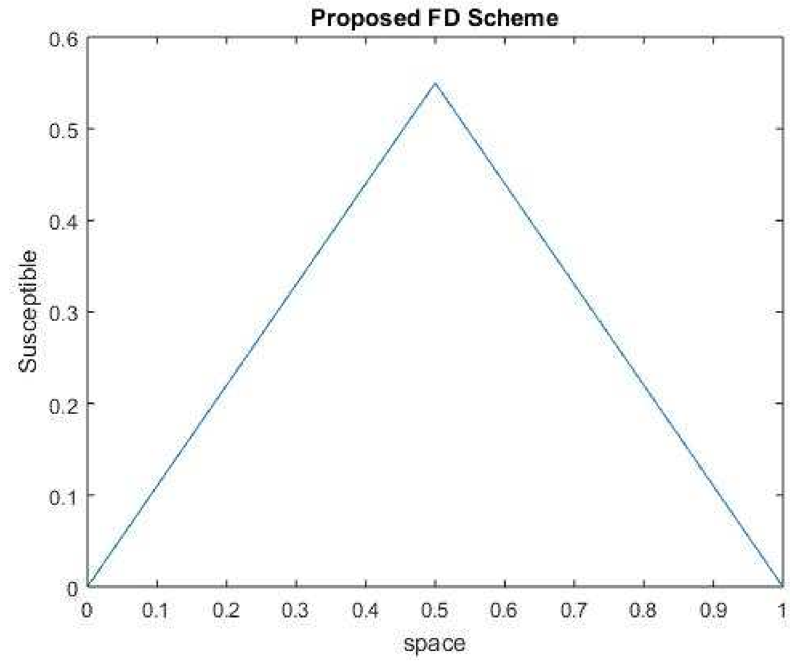

Figure 5(a)

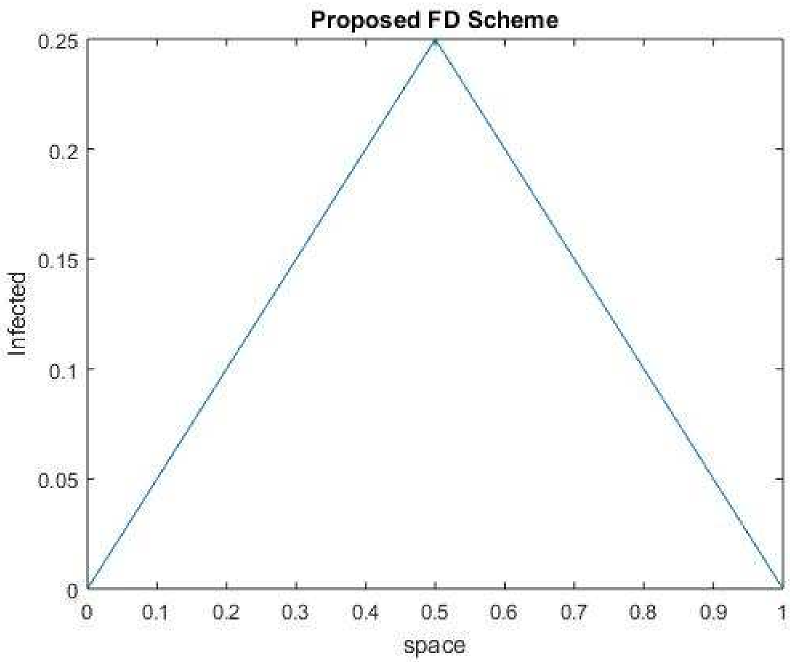

Figure 5(b)

Figure 5:Figures 5(a) – 5(b) reveals the graphs of susceptible and infected individuals for initial distributions.

Figure 5 reflects that the concentration of susceptible and infected individuals is maximum at the middle of the interval [0,1].

4.3.1. Disease Free Equilibrium:

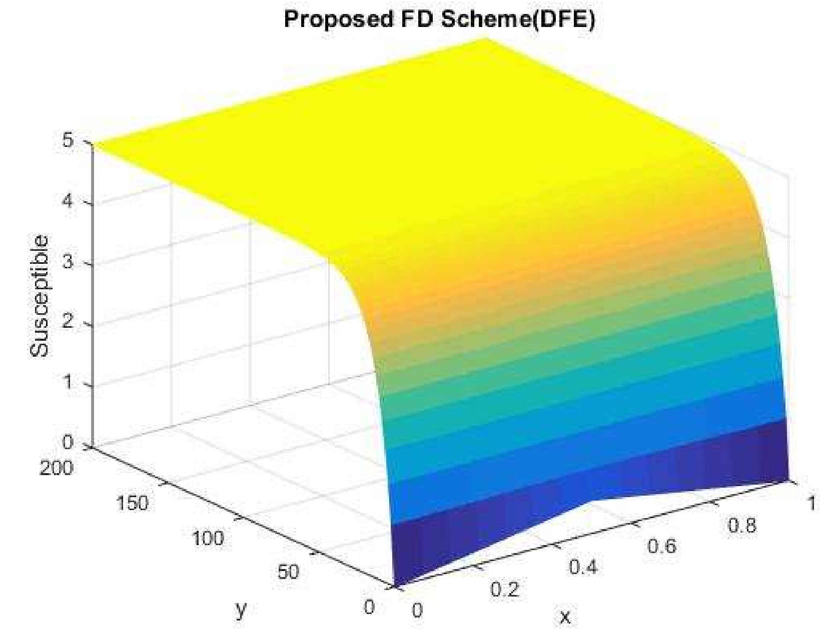

Figure 6(a)

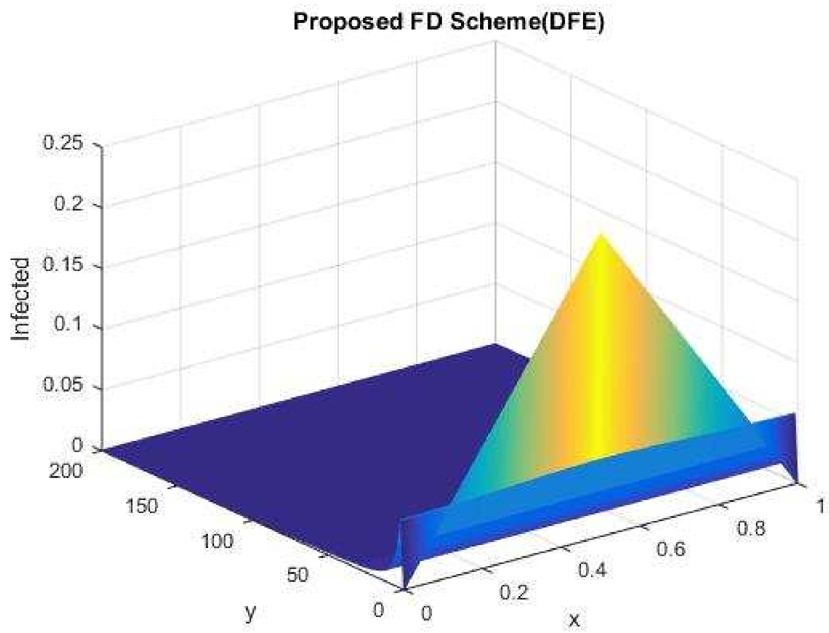

Figure 6(b)

Figure 6:Figures 6(a) – 6(b) represent the graphs of susceptible and infected individuals for disease free equilibrium using proposed FD scheme at $h = 0.001$, $\eta_1 = 2 \times 10^{-5}$ and $\eta_2 = 10^{-6}$

Figure 6 verifies the statement of theorem 3.3.1 as proposed FD scheme shows the positive behavior for both susceptible and infected individuals. Also proposed FD scheme converges to the disease free equilibrium point $E_{DFE}(S_{DFE}, I_{DFE}) = E_{DFE}(\lambda/\pi, 0) = E_{DFE}(5,0)$.

4.3.2. Endemic Equilibrium:

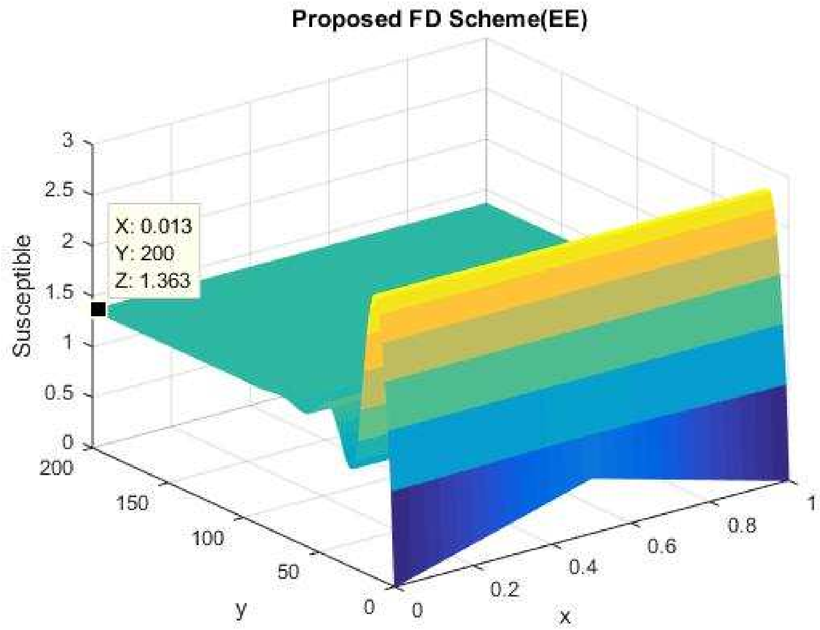

Figure 7(a)

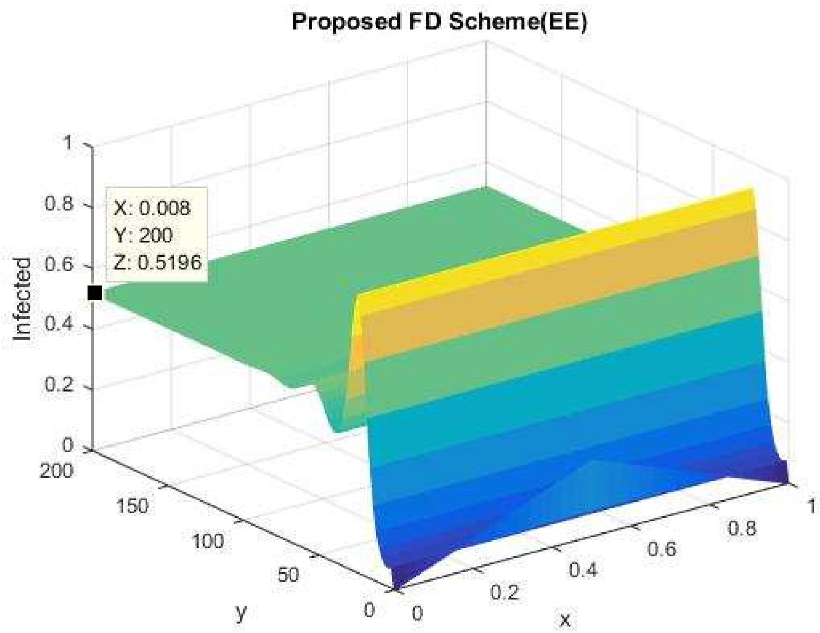

Figure 7(b)

Figure 7:Figures 7(a) – 7(b) represent the graphs of susceptible and infected individuals for disease free equilibrium using proposed FD scheme at $h = 0.001$, $\eta_1 = 2 \times 10^{-5}$ and $\eta_2 = 10^{-6}$.

The verification of the theorem 3.3.1 is shown in the figure 7 as the proposed FD scheme preserves positivity property. Also proposed scheme converges to the endemic equilibrium point $E_{EE}(S_{EE}, I_{EE}) = E_{EE}(1.3625, 0.5196)$ as shown in figure 7(a) and 7(b).

5. CONCLUSION

This paper is concerned about the numerical solution of reaction-diffusion epidemic model with specific nonlinear incidence rate. In this article, we developed a structural preserving implicit finite difference scheme for SIR reaction-diffusion epidemic model with specific nonlinear incidence rate. The proposed FD scheme is unconditionally dynamically consistent with positivity property. Also proposed FD scheme unconditionally converges to the true steady states (equilibrium points) of the continuous model. We also presented the convergence analysis of the proposed FD scheme and proved that proposed NSFD scheme is unconditionally stable and consistent with the help of Von Neumann stability analysis and Tayler series expansion respectively. We also proved in theorem 3.1.1 by induction that proposed FD scheme preserves positivity. The results are compared with the well-known backward Euler implicit FD scheme and Crank Nicolson implicit FD scheme. Both classical FD schemes fail to preserve positivity property, give non-physical behavior and converge to false steady states. On the other side proposed NSFD scheme is unconditionally convergent to true steady states. Simulations are done to verify all the claims of the proposed FD scheme.

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