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Error Bounds for a Numerical Scheme with Reduced Slope Evaluations

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ABSTRACT

In this paper, a new scheme of Runge-Kutta (RK) type has been discussed which utilizes only two slope evaluations per integration step while maintaining the third order accuracy as shown by the derived principal term of local truncation error. Characteristic stability polynomial is presented followed by the error analysis of the scheme. Error bounds in terms of Lotkin's bounds of the scheme have been derived and compared with the error bounds with existing standard schemes having similar number of slope evaluations as that of the scheme under consideration. Couples of numerical examples with varying nature are presented to test the performance of the developed scheme against some of the standard schemes having same number of slope evaluations per integration step.

KEYS WORDS: Initial value problems, Runge-Kutta scheme, autonomous and non-autonomous differential equations, Zero-stability.

1. INTRODUCTION

Ordinary Differential Equations (ODEs) have played a vital role in mathematical and biological sciences. Mathematical models based upon ordinary differential equations are very much important in various field of science including Biology (DNA molecules or biosynthesis phospholipids), Physics (Simple Pendulum), Chemistry (chemical reaction kinetics), Medicine (Pharmaceutical Drug Design), Population Dynamics (Verhulst-Pearl model), Engineering (beats of a vibrating system) and many more [1-5]. Mathematical modeling of ODEs is widely used in physical applications of above different kinds of indispensable areas. There are many problems of engineering and physical science which can be formulated into ordinary differential equations satisfying certain conditions (initial and/or boundary). If these conditions are prescribed for one and only point x_0 then such a problem together with the condition is known as an Initial Value Problem (IVP) as described in [6-8]. Main focus of the present paper is upon solving IVPs numerically with reduced slope evaluations required per integration step having discussion and derivation of the error bounds on the constant step size used for the numerical scheme under consideration.

Analytically, the solution to an initial value problem means finding an explicit expression for the unknown function $y(x)$. But analytical schemes are applicable only for selected class of IVPs mostly linear ones and a very few nonlinear IVPs. In cases when closed form solution does not exist, one has to go for the approximate solution using some numerical schemes as discussed in [9-10].

Various numerical schemes in past with different characteristics have been presented to solve such IVPs. In [11-15], authors have attempted to improve the order of accuracy of the existing standard linear RK type schemes whereas others in [16-18] have proposed new linear schemes with different characteristics. There is yet another group of scholars who have developed nonlinear schemes to solve those IVPs having rational solutions with some sort of singularity in them for which standard linear RK schemes do not perform well as can be consulted with [19-20]. Explicit and implicit or semi-implicit numerical schemes have also been derived to serve the purpose as shown in [21].

Consider an initial value problem in the form of

$$\frac{dy}{dx} = f\left(x, y(x)\right); \; y(x_0) = y_0, \quad x \in [x_0, x_n] \tag{1}$$

where uniqueness of the solution of (1) has been assumed.

One of the most common numerical schemes for solving the equation (1) is the standard classical RK scheme as explained in [7]. Most efforts to increase the order of RK methods have been accomplished by increasing the number of terms of Taylor's series thus the number of function evaluations per integration step. Three number of function evaluations are required per integration step in the classical third order RK method. Many authors have attempted to increase the efficiency of RK method with lower number of function evaluations required as detailed in [22-26].

As a result, a third order RK type scheme has been proposed in [27] which requires only two function evaluations per integration step to solve only autonomous type of IVPs. In order to get a numerical scheme applicable for both autonomous and non-autonomous IVPs, an improved RK type numerical scheme was proposed in [25] which also employs two function evaluations per step while maintaining third order accuracy but the paper has not offered error analysis and bound on the step length (h) of the scheme. Taking inspiration from this research work, the present paper not only offers derivation of the scheme but also contains analysis of its local truncation error and corresponding error bounds.

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The structure of the present paper has been organized as follows. In section 2, we presented the derivation of the third order improved RK scheme with two stages using the Taylor's series expansion. In section 3, the stability region of the scheme is discussed followed by the analysis of its Local Truncations Error (LTE) in section 4. Errors bounds of the scheme are further analyzed in section 5 whereas the results so obtained are discussed in the section 6 followed by section 7 for conclusion.

2. Derivation of the Improved RK Scheme

Consider the following structure of the proposed numerical scheme:

$$y_{n+1} = y_n + h(b_1 k_1 - b_{-1} k_{-1} + b_2 (k_2 - k_{-2})) \tag{2}$$

$$k_1 = f\left(x_n\ , y_n\right)$$

$$k_{-1} = f\left(x_{n-1}\ , y_{n-1}\right)$$

$$k_2 = f\left(x_n + c_2 h\ ,\ y_n + a_{21} k_1 h\right)$$

$$k_{-2} = f\left(x_{n-1} + c_2 h\ ,\ y_{n-1} + a_{21} k_{-1} h\right)$$

with the assumptions that $c_2 = a_{21}$ because $c_i = \sum_j^{i-1} a_{ij}$ and $c_2 \in (0,1]$.

In the above equations, $c_2 = a_{21}, b_1, b_{-1},\ b_2$ are the constants to be determined. Here k_1, k_{-1}, k_2, k_{-2} are the slopes to be Taylor expanded to get the proposed scheme in the following way:

$$k_1 = f$$

$$\begin{aligned} k_{-1} = {} & f + \left(-f_x - f_y f\right)h \\ & + \frac{h^2}{2!}\left[f_{xx} + 2 f_{xy} f + f_{yy} f + \quad f_y^2 f + f_y f_x\right] \\ & + \frac{h^3}{3!}\begin{bmatrix} -f_{xxx} - 3 f_{xxy} f - 3 f_{xyy} f^2 - f_{yyy} f^3 - 4 f^2 f_{yy} f_y - f_y^3 f \\ -f_x f_y^2 - 5 f_{xy} f_y f - 3 f_{yy} f_x f - f_{xx} f_y - 3 f_{xy} f_x \end{bmatrix} + O\left(h^4\right) \end{aligned}$$

$$\begin{aligned} k_2 = {} & f + \left(f_x c_2 + f_y c_2 f\right)h + \left(\frac{1}{2} f_{y,y} f^2 c_2^2 + f_{x,y} f c_2^2 + \frac{1}{2} f_{x,x} c_2^2\right) h^2 + \\ & \left(\frac{1}{6} f_{y,y,y} f^3 c_2^3 + \frac{1}{2} f_{x,y,y} f^2 c_2^3 + \frac{1}{2} f_{x,x,y} f c_2^3 + \frac{1}{6} f_{x,x,x} c_2^3\right) h^3 + O\left(h^4\right) \end{aligned}$$

$$\begin{aligned} k_{-2} = {} & f + \left(-f_x - f_y f + c_2 f_y f + c_2 f_x\right)h \\ & + \frac{h^2}{2!}\begin{bmatrix} f_{yy} f^2 c_2^2 + 2 f_{xy} f c_2^2 - 2 f_{yy} f^2 c_2 - 4 f_{xy} f c_2 + f^2 f_{yy} + f_{xx} c_2^2 \\ -2 f_y^2 f c_2 - 2 f_x f_y c_2 + f_y^2 f + 2 f_{xy} f - 2 f_{xx} c_2 + f_x f_y + f_{xx} \end{bmatrix} + \\ & + \frac{h^3}{3!}\begin{bmatrix} -3 f_{xxy}^3 f - 3 f^2 f_{xyy} - 3 f_{xy} f_x - f^3 f_{yyy} - f_y^3 f - f_y^2 f_x - f_{xx} f_y - 6 f^2 f_{yy} f_y c_2 \\ -6 f_{yy} f_x f c_2 - 6 f_{xy} f_y f c_2 - 3 f^3 f_{yyy} c_2^2 + 3 f^3 f_{yyy} c_2 - 9 f^2 f_{xyy} c_2^2 + f_{yy} f_x f c_2 \\ +15 f_{xy} f_y f c_2 + 9 f^2 f_{xyy} c_2 - 15 f_{xy} f_y f - 4 f^2 f_{yy} f_y - 3 f_{yy} f_x f - f_{xxx} \\ +3 f^2 f_y f_{yy} c_2 + 3 f_y^3 f c_2 + 3 f_y^2 f_x c_2 + 3 f_{xx} f_y c_2 - 6 f_{xy} f_x c_2^2 - 9 f f_{xxy}^2 c_2^2 + 9 f_{xy} f_x c_2 \\ +9 f_{xxy} f c_2 + 3 f^2 f_{xyy} c_2^3 + 3 f f_{xxy} c_2^3 + f^3 f_{yyy} c_2^3 - 3 f_{xxx} c_2^2 + 3 f_{xxx} c_2 + f_{xxx} c_2^3 \end{bmatrix} + O\left(h^4\right) \end{aligned}$$

$$y_{n+1} = y_n + hf\left(b_1 - b_{-1}\right) + h^2\left(f_x + f_y f\right)\left(b_{-1} + b_2\right) + \frac{1}{2}h^3\begin{bmatrix} f_{xx}(-b_{-1} + 2b_2c_2 - b_2) + 2f_{xy}f(-2b_{-1} + 4b_2c_2 - 2b_2) \\ + f_{yy}f^2(-b_{-1} + 2b_2c_2 - b_2) + f_y{}^2 f(-b_{-1} + 2b_2c_2 - b_2) \\ + f_y f_x(-b_{-1} + 2b_2c_2 - b_2) \end{bmatrix}$$

$$\frac{h^4}{6}\begin{bmatrix} f_{xxy}f\left(3b_2 + 9b_2c_2^2 - 9b_2c_2 + 3b_{-1}\right) + f^2 f_{xyy}\left(3b_2 + 9b_2c_2^2 - 9b_2c_2 + 3b_{-1}\right) + f_{xy}f_x\left(3b_2 + 6b_2c_2^2 - 9b_2c_2 + 3b_{-1}\right) \\ + f_{yyy}^3 f\left(b_2 + 3b_2c_2^2 - 3b_2c_2 + b_{-1}\right) + f\, f_y^3\left(b_2 - 3b_2c_2 + b_{-1}\right) + f_y^2 f_x\left(b_2 - 3b_2c_2 + b_{-1}\right) + f_{xx}f_y\left(b_2 - 3b_2c_2 + b_{-1}\right) \\ f_{yy}f_y f^2\left(6b_2c_2^2 + 4b_2 - 12b_2c_2 + 4b_{-1}\right) + f_{yy}f_x f\left(6b_2c_2^2 + 9b_2c_2 + 3b_2 + 3b_{-1}\right) \\ + f_{xy}f_y f\left(6b_2c_2^2 - 15b_2c_2 + 5b_2 + 5b_{-1}\right) + f_{xxx}\left(3b_2c_2^2 - 3b_2c_2 + b_2 + b_{-1}\right) \end{bmatrix} \quad (3)$$

Generally, the Taylor's series for a function $y\left(x_n + h\right)$ is as follows:

$$y\left(x_n + h\right) = y\left(x_n\right) + hf + \left(\frac{1}{2}f_x + \frac{1}{2}f_y f\right)h^2 +$$
$$\left(\frac{1}{6}f_{x,x} + \frac{1}{3}f_{x,y}f + \frac{1}{6}f^2 f_{y,y} + \frac{1}{6}f_y^2 f + \frac{1}{6}f_x f_y\right)h^3 + \quad (4)$$
$$\begin{pmatrix} \frac{1}{24}f_{x,x,x} + \frac{1}{8}f_{x,x,y}f + \frac{1}{8}f_{x,y,y}f^2 + \frac{5}{24}f_y f f_{x,y} + \frac{1}{8}f_x f_{x,y} + \frac{1}{24}f_{y,y,y}f^3 + \\ \frac{1}{6}f_y f^2 f_{y,y} + \frac{1}{8}f_x f f_{y,y} + \frac{1}{24}f_y^3 f + \frac{1}{24}f_x f_y^2 + \frac{1}{24}f_y f_{x,x} \end{pmatrix}h^4 + O\left(h^5\right)$$

Comparing the equations (3) and (4) up to h^3 terms, the following order conditions are obtained:

$$b_1 - b_{-1} = 1,\ b_{-1} + b_2 = \frac{1}{2},\ -b_{-1} + 2b_2c_2 - b_2 = \frac{1}{6} \quad (5)$$

After solving this system of nonlinear algebraic equations, we have the following general structure where c_2 be a free parameter:

$$b_{-1} = \frac{1}{12}\frac{-5 + 6c_2}{c_2},\ b_1 = \frac{1}{12}\frac{-5 + 18c_2}{c_2},\ b_2 = \frac{5}{12c_2} \quad (6)$$

After trying various values of the free parameter c_2, we have come up with following numerical scheme which employs two slope evaluations per integration step while maintaining the third order accuracy as shown by the local truncation error in the next section:

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{h}{4}\left[k_1 + 3k_{-1} + 5\left(k_2 - k_{-2}\right)\right] \\ k_1 &= f\left(x_n\ , y_n\right);\quad k_{-1} = f\left(x_{n-1}\ , y_{n-1}\right) \\ k_2 &= f\left(x_n + \frac{1}{3}h\ , y_n + \frac{1}{3}k_1 h\right);\ k_{-2} = f\left(x_{n-1} + \frac{1}{3}h\ , y_{n-1} + \frac{1}{3}k_{-1}h\right) \end{aligned}\right\} \quad (7)$$

3. Error Analysis

For getting the expression of local truncation error of the proposed numerical scheme, a functional related to the scheme is considered:

$$L\left(z\left(x\right), x\right) = z\left(x_{n+1}\right) - \text{Improved RK Scheme shown by } (7)$$

where $z\left(x\right)$ is a function arbitrary in nature and can be differentiated as many times as required on $\left[x_0, x_n\right]$. Having Taylor expanded the above equation about x, it is observed that the all terms up to h^3 have been cancelled and thus the local truncation error comes out to be as follows:

$$LTE = \frac{h^4}{72}\begin{bmatrix} 7\left(f_{xxx} + f_{yyy} f^3\right) + 21\left(f_{xxy} f + f_{xyy} f^2\right) + \\ 12\left(f_y^3 f + f_x f_y^2 + f_{xx} f_y\right) + \\ 26\left(f_{yy} f_x f + f_{xy} f_x\right) + 50 f_{xy} f_y f + 38 f^2 f_{yy} f_y \end{bmatrix} + O\left(h^5\right) \tag{8}$$

Further, being at least third order accurate the developed numerical scheme (7) is also **consistent**.

4. Stability Analysis

The stability of the developed numerical scheme in (7) is checked using it to the Dahlquist's test problem [28] as follows:

$$y'(x) = \lambda y(x); \ \mathrm{Re}(\lambda) < 0 \tag{9}$$

The slopes involved in the scheme become

$$k_1 = \lambda y_n,\ k_{-1} = \lambda y_{n-1}$$

$$k_2 = \lambda\left(1 + \frac{h\lambda}{3}\right) y_n,\ k_{-2} = \lambda\left(1 + \frac{h\lambda}{3}\right) y_{n-1}$$

After simplification, we get the following stability polynomial

$$\rho(\omega, z) = \omega^2 + p(z)\omega + q(z) \tag{10}$$

where $p(z) = -\frac{1}{12}\left(5z^2 + 18z + 12\right)$, $q(z) = \frac{1}{12}\left(5z^2 + 6z\right)$, and $z = h\lambda$. Further, ω_1 and ω_2 are said to be the zeros of the stability polynomial (10).

For analyzing stability and obtaining stability region of the developed numerical scheme, we employ **Schur – Cohn stability criterion** discussed in [29]. For this purpose, we define:

$$\overline{\rho}(\omega, z) = q'\omega^2 + p'\omega + 1 \text{ and } \overline{\rho}_1(\omega, z) = \frac{1}{\omega}\left[\overline{\rho}(0, z)\rho(\omega, z) - \rho(0, z)\overline{\rho}(\omega, z)\right]$$

where p' and q' are complex conjugates of p and q respectively. Here, $\overline{\rho}_1(\omega, z)$ is at least of first degree polynomial as shown below:

$$\overline{\rho}_1(\omega, z) = \frac{1}{\omega}\left[\omega^2 + p\omega + q - q\left(q'\omega^2 + p'\omega + 1\right)\right] = \omega + p - qq'\omega - qp'$$

By Schur-Cohn theorem, $|\omega_1| < 1$ and $|\omega_2| < 1$ if and only if $|q| < 1$ and $|p - qp'| + |q|^2 < 1$.

Region of the stability for the developed numerical scheme is sketched (unshaded region) below along-with its interval of stability:

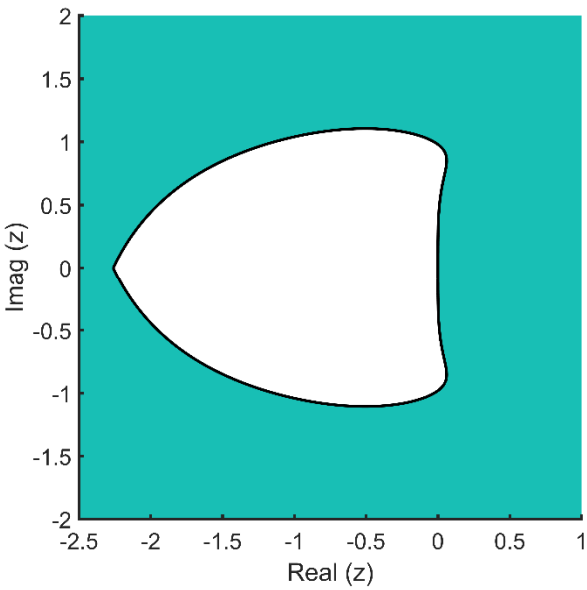

Figure 1. Region of stability with interval of stability being $(-2.248, 0.055)$ in the Complex Plane

This completes the proof of stability of the developed numerical scheme. Moreover, the first characteristic polynomial $\rho(z)=0$, has two zeros, that is, $\omega = 0,1$; it can be claimed that the scheme is **zero – stable**. Further, depending upon the above discussion of consistency and stability the convergence of the numerical scheme is analyzed in the following way:

Theorem (Dahlquist's Equivalence Theorem)
As reported in [30]; "For a linear multistep numerical scheme consistent with an ordinary differential equation $y'(x)=f(x,y(x))$; where $f(x,y(x))$ is assumed to satisfy the Lipchitz condition, zero-stability and consistency are necessary and sufficient conditions for convergence."
Mathematically,

$$\text{Consistency+Zero-Stability} \Leftrightarrow \text{Convergence}$$

This theorem guarantees for the developed scheme to be **convergent**.
As far as order of accuracy of the scheme is concerned; it has been revealed while deriving the scheme that it takes three order conditions (5) and utilizes Taylor's expansion up to the term containing h^3. This clearly shows that global error involved in the method is of order **three**, that is, global truncation error $= O(h^3)$ hence the order of the scheme.

5. Propagation of Errors

It is a common fact that numerical solution of an ordinary differential equation contains round off (uncontrollable) and truncation (discretization) errors where truncation error is generally of two types, that is; local and global truncation errors, which remain under control of the analyst.
Detailed study of magnitude and characteristics of truncation error is important to accept any new devised iterative method. In order to be of some use, an iterative must discuss error bounds it contains as claimed in [31]. The developed numerical scheme (7) can easily be compared with Taylor series expansion of the form:

$$\begin{aligned} y(x_n+h) &= y_n + hf + \frac{h^2}{2}\left(f_x + f_y f\right) + \frac{h^3}{6}\left(f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)^2 f + f_y f_x\right) \\ &+ \frac{h^4}{24}\begin{pmatrix} f_{xxx} + 3f_{xxy}f + 3f_{xyy}f^2 + f_{yyy}f^3 + 4f^2 f_{yy} f_y \\ + f_y^3 f + f_x f_y^2 + 5f_{xy}f_y f + 3f_{yy}f_x f + f_{xx}f_y + 3f_{xy}f \end{pmatrix} + R_n \end{aligned} \tag{11}$$

where $R_n = \dfrac{y^{(n+1)}(\eta)}{(n+1)!}h^{n+1},\ x_k < \eta < x_{k+1}$

The terms containing h^4 have been truncated while developing the modified iterative method resulting the bound of local truncation error given by:

$$LTE \le \max_{k=\{01,2,\cdots,N\}} \frac{h^4}{4!} y^{(iv)}(\eta) \text{ for } x_k < \eta < x_{k+1}$$

According to above inequality, we find that local truncation error is proportional to the power 4 of the step size and fourth derivative of the given differential equation. Likewise, global truncation error will be $GTE = O(h^3)$. It implies that halving the step size will decrease the error by a factor of about $1/8$. In order to prove it in general sense; error bound for local truncation error (LTE) is computed using Lotkin's Error Bounds discussed by Lotkin in [32]. From (8), the bound for the local truncation error is obtained as:

$$\left|\psi(x_k, y_n)h^4\right| < \frac{29}{9}h^4 P^3 Q \tag{12}$$

where $\psi(x_k, y_n)$ is known as *Principal Error Function* for the proposed method and *P, Q* are positive constants given by Lotkin as:

$$\left|f(x,y)\right| < Q \text{ and } \left|\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right| < \frac{P^{i+j}}{Q^{j-1}};\ (i+j) \le \text{order of the scheme}$$

Following table shows error bounds, number of function evaluations and order of accuracy of some numerical schemes compared with that of the developed scheme (7). The Table 1 reveals strength of the developed scheme in connection with order of accuracy in particular. The table also shows that the third order standard linear Ralston's scheme will take comparatively larger step size with three function evaluations per integration step whereas the scheme developed exploits only two.

Table 1. Error Bounds

Method	Error Bound on LTE	Function Evaluations	Order of Accuracy
Improved Euler	$\left\|\psi\left(t_n, y_n\right)h^3\right\| < \frac{2}{3}h^3P^2Q$	TWO	2
Midpoint Euler	$\left\|\psi\left(t_n, y_n\right)h^3\right\| < \frac{1}{2}h^3P^2Q$	TWO	2
Ralston	$\left\|\psi\left(t_n, y_n\right)h^3\right\| < \frac{5}{12}h^3P^2Q$	TWO	2
Ralston	$\left\|\psi\left(t_n, y_n\right)h^4\right\| < 0.1111h^4P^3Q$	THREE	3
Proposed	$\left\|\psi\left(t_n, y_n\right)h^4\right\| < \frac{29}{9}h^4P^3Q$	**TWO**	**3**

6. RESULTS AND DISCUSSION

For testing the developed numerical scheme, couples of initial value problems of varying nature are considered from the literature. Standard linear numerical schemes having same number of slope evaluations per integration step are selected for comparison with the developed scheme. To serve the purpose, maximum absolute error along the integration interval $\left(E = \max_{n=1,2,\cdots,N}\left|y\left(x_n\right) - y_n\right|\right)$, final absolute global error $\left(E\left(t = x_n\right) = \left|y\left(x = N\right) - y_N\right|\right)$ and CPU time values have been tabulated for all the schemes under consideration. Each data cell in every table of the numerical experiments lists the maximum absolute error, final absolute global error and the CPU time values from top to bottom order. In all types of initial value problems under consideration in the present paper, the developed scheme proposed here yields smaller errors in comparison with other schemes having same order of local accuracy. The graphs for this purpose tell the similar sort of story even though considerably small step size is chosen to solve the underlying IVP as depicted in the numerical experiments discussed below.

Example 1. In this first problem, a linear initial value problem is chosen which is given as:

$$\frac{dy}{dx} = x + y,\ y\left(0\right) = 1$$

whereas its exact analytical solution is given by:

$$y\left(x\right) = -x - 1 + 2e^x$$

It is observed from the Table 2 that with the increasing number of integration steps, maximum absolute error on the integration interval $\left[0,1\right]$ and the final absolute global error are decreasing for each numerical scheme under consideration with the developed scheme (7) having the smallest errors among all. CPU time values exhibit similar type of trend for almost every scheme in the table.

Table 2. Errors and CPU values for Example 1

Scheme/NI	64	128	256	512	1024
Proposed	3.3760e-06 3.3760e-06 0.0000e+00	4.2703e-07 4.2703e-07 0.0000e+00	5.3693e-08 5.3693e-08 0.0000e+00	6.7313e-09 6.7313e-09 0.0000e+00	8.4264e-10 8.4264e-10 0.0000e+00
Heun's	2.1863e-04 2.1863e-04 0.0000e+00	5.4980e-05 5.4980e-05 0.0000e+00	1.3785e-05 1.3785e-05 0.0000e+00	3.4514e-06 3.4514e-06 1.5625e-02	8.6349e-07 8.6349e-07 1.5625e-02
Ralston	2.1863e-04 2.1863e-04 0.0000e+00	5.4980e-05 5.4980e-05 0.0000e+00	1.3785e-05 1.3785e-05 0.0000e+00	3.4514e-06 3.4514e-06 0.0000e+00	8.6349e-07 8.6349e-07 0.0000e+00

Example 2. Once again, a linear initial value problem is considered but it has the slope with a transcendental function given on the right:

$$\frac{dy}{dx} = 6\sin\left(2x\right) - 20y\left(x\right),\ y\left(0\right) = 1$$

Its exact analytical solution is given by:

$$y\left(x\right) = -\frac{3}{101}\cos\left(2x\right) + \frac{30}{101}\sin\left(2x\right) + \frac{104}{101}e^{-20x}$$

It can be seen from the Table 3 that with the increasing number of integration steps, maximum absolute error on the integration interval $\left[0,1\right]$ and the final absolute global error are decreasing for each numerical scheme under consideration with the

developed scheme (7) having the smallest errors among all. CPU time values exhibit similar type of trend for almost every scheme in the table.

Table 3. Errors and CPU values for Example 2

Scheme/NI	64	128	256	512	1024
Proposed	1.9368e-03	2.4081e-04	3.0074e-05	3.7624e-06	4.7041e-07
	4.2495e-08	6.1629e-09	8.3113e-10	1.0792e-10	1.3750e-11
	6.2500e-02	0.0000e+00	0.0000e+00	0.0000e+00	1.0469e+00
Heun's	7.8433e-03	1.7335e-03	4.0882e-04	9.9209e-05	2.4442e-05
	8.3179e-05	1.9013e-05	4.5582e-06	1.1166e-06	2.7639e-07
	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
Ralston	7.8442e-03	1.7338e-03	4.0888e-04	9.9224e-05	2.4445e-05
	5.5931e-05	1.2780e-05	3.0633e-06	7.5036e-07	1.8571e-07
	1.5625e-02	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00

Example 3. In this example, a linear initial value problem is under discussion whose exact solution is not available in terms of elementary mathematical functions:

$$\frac{dy}{dx} = 2xy(x) - 1, \quad y(0) = 1$$

The exact solution consists of a special function called the Gauss error function [Temme, N. M, 2010]

$$y(x) = \left(-\frac{1}{2}\sqrt{\pi}\, erf(x) + 1 \right) e^{x^2}$$

It can be seen from the Table 4 that with the increasing number of integration steps, maximum absolute error on the integration interval $[0,1]$ and the final absolute global error are decreasing for each numerical scheme under consideration with the developed scheme (7) having the smallest errors among all. CPU time values exhibit similar type of trend for almost every scheme in the table.

Table 4. Errors and CPU values for Example 3

Scheme/NI	64	128	256	512	1024
Proposed	8.2727e-06	1.0554e-06	1.3326e-07	1.6741e-08	2.0978e-09
	8.2727e-06	1.0554e-06	1.3326e-07	1.6741e-08	2.0978e-09
	0.0000e+00	0.0000e+00	3.7500e-01	6.7188e-01	1.1406e+00
Heun's	1.6085e-04	4.0053e-05	9.9932e-06	2.4958e-06	6.2363e-07
	1.6085e-04	4.0053e-05	9.9932e-06	2.4958e-06	6.2363e-07
	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
Ralston	9.2021e-05	2.3037e-05	5.7634e-06	1.4414e-06	3.6040e-07
	9.2021e-05	2.3037e-05	5.7634e-06	1.4414e-06	3.6040e-07
	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00

Example 4. Here, a nonlinear autonomous initial value problem is under consideration:

$$\frac{dy}{dx} = y(x)(1 - y(x)), \quad y(0) = 0.5$$

The exact solution of which is provided by: $y(x) = \dfrac{1}{1 + e^{-x}}$

It can be seen from the Table 5 that with the increasing number of integration steps, maximum absolute error on the integration interval $[0,1]$ and the final absolute global error are decreasing for each numerical scheme under consideration with the developed scheme (7) having the smallest errors among all. CPU time values exhibit similar type of trend for almost every scheme in the table. Further, the developed numerical scheme is also tested with considerably fewer number of integration steps (NI=20) against other methods but still found to be better in terms of absolute errors as shown by the Figure 2.

Table 5. Errors and CPU values for Example 4

Scheme/NI	64	128	256	512	1024
Proposed	3.8438e-08	4.8357e-09	6.0639e-10	7.5920e-11	9.4965e-12
	3.8438e-08	4.8357e-09	6.0639e-10	7.5920e-11	9.4965e-12
	7.8125e-02	1.2500e-01	4.8438e-01	5.1563e-01	1.0000e+00
Heun's	2.4671e-06	6.1522e-07	1.5361e-07	3.8378e-08	9.5915e-09
	2.4671e-06	6.1522e-07	1.5361e-07	3.8378e-08	9.5915e-09
	1.5625e-02	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
Ralston	6.0860e-07	1.5184e-07	3.7923e-08	9.4758e-09	2.3683e-09
	6.0860e-07	1.5184e-07	3.7923e-08	9.4758e-09	2.3683e-09
	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00

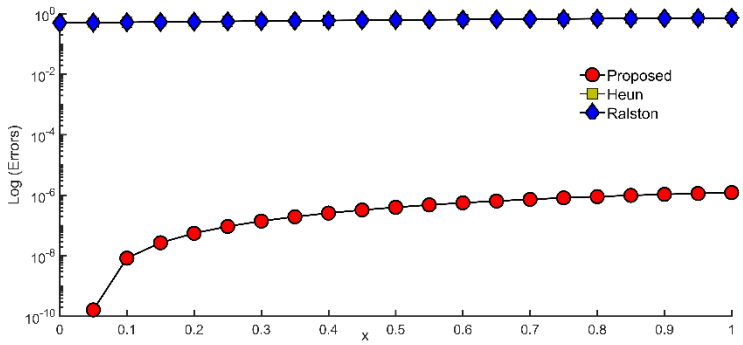

Figure 2. Absolute Errors with NI=20 for Example 4

Example 5. Finally, a nonlinear initial value problem with no solution in closed-form has been selected as follows:

$$\frac{d}{dx}y(x)=x^2-y(x)^2,\ \ y(0)=0$$

whose solution has been obtained using Maple symbolic environment as shown below:

$$y(x)=\begin{cases} 0 & x=0 \\ \dfrac{x\left(\mathrm{BesselI}\left(-\frac{3}{4},\frac{1}{2}x^2\right)\sqrt{2}\pi-2\mathrm{BesselK}\left(\frac{3}{4},\frac{1}{2}x^2\right)\right)}{\sqrt{2}\mathrm{BesselI}\left(\frac{1}{4},\frac{1}{2}x^2\right)+2\mathrm{BesselK}\left(\frac{1}{4},\frac{1}{2}x^2\right)} & \text{otherwise} \end{cases}$$

where $\mathrm{BesselI}(\nu,z)$ and $\mathrm{BesselK}(\nu,z)$ are the Bessel functions of the first and second kind respectively.

It can be seen from the Table 6 that with the increasing number of integration steps, maximum absolute error on the integration interval $[0,1]$ and the final absolute global error are decreasing for each numerical scheme under consideration with the developed scheme (7) having the smallest errors among all. CPU time values exhibit similar type of trend for almost every scheme in the table. Further, the developed numerical scheme is also tested with considerably fewer number of integration steps (NI=20) against other methods but still found to be better in terms of absolute errors as shown by the Figure 3.

Table 6. Errors and CPU values for Example 5

Scheme/NI	64	128	256	512	1024
Proposed	1.0483e-06	1.3285e-07	1.6720e-08	2.0972e-09	2.6259e-10
	1.0483e-06	1.3285e-07	1.6720e-08	2.0972e-09	2.6259e-10
	0.0000e+00	0.0000e+00	5.6250e-01	6.5625e-01	9.3750e-01
Heun's	3.7620e-05	9.3566e-06	2.3331e-06	5.8251e-07	1.4553e-07
	3.7620e-05	9.3566e-06	2.3331e-06	5.8251e-07	1.4553e-07
	0.0000e+00	0.0000e+00	4.6875e-02	0.0000e+00	0.0000e+00
Ralston	8.3089e-06	2.0616e-06	5.1343e-07	1.2811e-07	3.1996e-08
	8.3089e-06	2.0616e-06	5.1343e-07	1.2811e-07	3.1996e-08
	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00

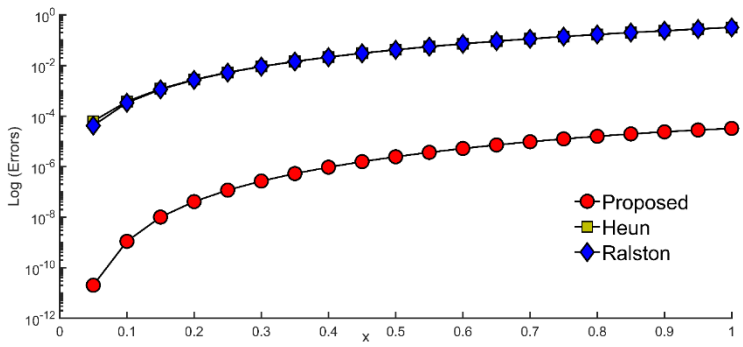

Figure 3. Absolute Errors with NI=20 for Example 5

7. CONCLUSION

The present work demonstrates the efficiency of an improved version of RK type scheme having third order accuracy especially in terms of number of slope evaluations per integration step and the error bound on the step length of the scheme. Errors produced by the presented scheme are much smaller than the errors given by other schemes taken for consideration. Although, the step length is taken to be as large as 0.05 but the curve of absolute errors of the scheme remains below the error curves of the other schemes considered for comparison. In addition to this, the presented scheme has error bound for which one may easily check the number of iterations required by the scheme before to actually employing it on an initial value problem. This has been shown that the scheme requires fewer number of integration steps in comparison with other schemes as given in the first table. In the future, the present research work is planned to be stretched for the derivation of a fourth order RK type scheme with three number of slope evaluations per integration step keeping its errors in control with the idea of the error bounds on its step length.

8. REFERENCES

[1] Butcher, John Charles. *Numerical methods for ordinary differential equations*. John Wiley & Sons, 2016.

[2] Saitoh, Saburou, and Yoshihiro Sawano. "Applications to Ordinary Differential Equations." *Theory of Reproducing Kernels and Applications*. Springer, Singapore, 2016. 217-230.

[3] Kanwal, Ram P. "Applications to Ordinary Differential Equations." *Linear Integral Equations*. Birkhäuser, New York, NY, 2013. 61-96.

[4] Cesari, Lamberto. *Optimization—theory and applications: problems with ordinary differential equations*. Vol. 17. Springer Science & Business Media, 2012.

[5] Chicone, Carmen. *Ordinary differential equations with applications*. Vol. 34. Springer Science & Business Media, 2006.

[6] Ahmad, Shair, and M. Rama Mohana Rao. *Theory of Ordinary Differential Equations: With Applications of Biology and Engineering*. Affiliated East-West Private Lmt., 1999.

[7] Chapra, Steven C., and Raymond P. Canale. *Numerical methods for engineers*. Vol. 2. New York: McGraw-Hill, 1998.

[8] Nakamura, Shoichiro. "Computational methods in engineering and science, with applications to fluid dynamics and nuclear systems." (1977).

[9] Qureshi, Sania, et al. "ON ERROR BOUND FOR LOCAL TRUNCATION ERROR OF AN EXPLICIT ITERATIVE ALGORITHM IN ORDINARY DIFFERENTIAL EQUATIONS." *Science International* 26.2 (2014).

[10] QURESHI, SANIA, ASIF ALI SHAIKH, and MUHAMMAD SALEEM CHANDIO. "A NEWITERATIVE INTEGRATOR FOR CAUCHY PROBLEMS." *Sindh University Research Journal-SURJ (Science Series)* 45.3 (2013).

[11] Ochoche, Abraham. "Improving the Improved Modified Euler Method for Better Performance on Autonomous Initial Value Problems." *Leonardo Journal of Sciences* 7.12 (2008): 57-66.

[12] Ochoche, Abraham. "Improving the modified Euler method." *Leonardo Journal of Sciences* 6.10 (2007): 1-8.

[13] Ochoche, Abraham, and Peter Ndajah. "Almost Runge-Kutta methods of orders up to five." *WSEAS Transactions on Mathematics* 10.5 (2011): 159-168.

[14] Ochoche, Abraham. "Improving the Improved Modified Euler Method for Better Performance on Autonomous Initial Value Problems." *Leonardo Journal of Sciences* 7.12 (2008): 57-66.

[15] Onumanyi, P., et al. "New linear mutlistep methods with continuous coefficients for first order initial value problems." *J. Nig. Math. Soc* 13.7 (1994): 37-51.

[16] Wazwaz, Abdul-Majid. "A new method for solving singular initial value problems in the second-order ordinary differential equations." *Applied Mathematics and computation* 128.1 (2002): 45-57.

[17] Gear, Charles William. "Hybrid methods for initial value problems in ordinary differential equations." *Journal of the Society for Industrial and Applied Mathematics, Series B: Numerical Analysis* 2.1 (1965): 69-86.

[18] Wang, Zhongcheng. "P-stable linear symmetric multistep methods for periodic initial-value problems." *Computer Physics Communications* 171.3 (2005): 162-174.

[19] Ramos, Higinio. "A non-standard explicit integration scheme for initial-value problems." *Applied Mathematics and Computation* 189.1 (2007): 710-718.

[20] Vigo-Aguiar, Jesús, and Higinio Ramos. "Variable stepsize implementation of multistep methods for y ″= f (x, y, y′)." *Journal of Computational and Applied Mathematics* 192.1 (2006): 114-131.

[21] Yahya, Nur Azila. "Semi-implicit two-step hybrid method with FSAL property for solving second-order ordinary differential equations." *INTERNATIONAL JOURNAL OF ADVANCED AND APPLIED SCIENCES* 4.6 (2017): 169-174.

[22] F. Rabiei and F. Ismail, "Numerical solution of ordinary differential equation using fifth-order Improved Runge-Kutta method," International Journal of Applied Applied Mathematics and Informatics, Submitted 2011.

[23] F. Rabiei and F. Ismail, "Fifth-order Improved Runge-Kutta method for solving ordinary differential equation," Proceeding of WSEAS Conference, Penang, Malaysia, ISBN: 978-1- 61804-039-8, 2011, pp. 129-133.

[24] F. Rabiei, F. Ismail, M. Suleiman, and N. Arifin, "Improved Runge-Kutta method for solving ordinary differential equation," Journal of Applied Mathematics, submitted 2011.

[25] F. Rabiei and F. Ismail, "New Improved Runge- Kutta method with reducing number of function evaluation," ASME Press (2011), ISSN:9780791859797, DOI: 10.1115/1.859797. paper14

[26] F. E. Udwadia and A. Farahani, "Accelerated Runge- Kutta methods," Discrit. Dynamic. Nature. Soci. 2008, doi:10.1155/2008/790619, 2008.

[27] Goeken, David, and Olin Johnson. "Runge–Kutta with higher order derivative approximations." *Applied numerical mathematics* 34.2-3 (2000): 207-218.

[28] Dahlquist, Germund G. "A special stability problem for linear multistep methods." *BIT Numerical Mathematics* 3.1 (1963): 27-43.

[29] Sah, Raaj Kumar. "A useful statement of some Schur-Cohn stability criteria for higher order discrete dynamic systems." *IEEE transactions on automatic control* 36.8 (1991): 988-989.

[30] Gautschi, Walter. *Numerical analysis.* Springer Science & Business Media, 2011.

[31] Akanbi, M. A. "Propagation of Errors in Euler Methods." *Archives of Applied Science Research* 2.2 (2010): 457-469.

[32] Lotkin, Max. "On the accuracy of Runge-Kutta's method." *Mathematics of Computation* 5.35 (1951): 128-133.