Local Analysis FGM Circular Cylindrical Shell with Free-Simply Support Boundary Conditions

Mohammad Setareh 1, Mohammad Reza Isvandzibaei 2

1,2 Department of Mechanical Engineering, Andimeshk Branch, Islamic Azad University, Andimeshk, Iran

ABSTRACT

Study of the vibration cylindrical shells is very important. Material properties are graded in the thickness direction of the shell according to volume fraction power law distribution. The objective is to study the natural frequencies, the influence of constituent volume fractions and the effects of boundary conditions on the natural frequencies of the FGM cylindrical shell. The study is carried out using third order shear deformation shell theory. The governing equations of motion of FGM cylindrical shells are derived based on shear deformation theory. Results are presented on the frequency characteristics and the effects of F-SS boundary conditions.

KEYWORDS: Analysis, FGM, Cylindrical shell.

INTRODUCTION

Understanding of vibration behavior of cylindrical shells is an important aspect for the successful applications of cylindrical shells.

Researches on free vibrations of cylindrical shells have been carried out extensively [1-5]. Recently, the present authors presented studies on the influence of boundary conditions on the frequencies of a multi-layered cylindrical shell [6]. In all the above works, different thin shell theories based on Love-hypothesis were used.

Vibration of cylindrical shells with ring support is considered by Loy and Lam [7]. The concept of functionally graded materials (FGMs) was first introduced in 1984 by a group of materials scientists in Japan [8-9] as a means of preparing thermal barrier materials. Since then, FGMs have attracted much interest as heat-shielding materials.

FGMs are made by combining different materials using power metallurgy methods [10]. They possess variations in constituent volume fractions that lead to continuous change in the composition, microstructure, porosity, etc., resulting in gradients in the mechanical and thermal properties [11-12]. Vibration study of FGM shell structures is important. However, study of the vibration of FGM shells with ring supports is limited.

The FGMs considered are composed of stainless steel and nickel where the volume fractions follow a power-law distribution. The study is carried out based on third order shear deformation shell theory. Studies are carried out for cylindrical shells with clamped-simply support (F-SS) boundary conditions. Results presented include the frequency characteristics of cylindrical shells, and the influence of boundary conditions. The present analysis is validated by comparing results with others in the literature.

1- FUNCTIONALLY GRADED MATERIAL

For the cylindrical shell made of FGM the material properties such as the modulus of elasticity $E$, Poisson ratio $\nu$ and the mass density $\rho$ are assumed to be functions of the volume fraction of the constituent materials when the coordinate axis across the shell thickness is denoted by $z$ and measured from the shell’s middle plane. The functional relationships between $E$, $\nu$ and $\rho$ with $z$ for a stainless steel and nickel FGM shell are assumed as [13].

\[ E = (E_1 - E_2) \left( \frac{2Z + h}{2h} \right)^n + E_2 \]  
\[ \nu = (\nu_1 - \nu_2) \left( \frac{2Z + h}{2h} \right)^n + \nu_2 \]  

*Corresponding Author: Mohammad Reza Isvandzibaei, Ph.D. Student in Mechanical Engineering, Islamic Azad University, Andimeshk, Iran. Email: esvandzebaei@yahoo.com
\[ \rho = (\rho_1 - \rho_2) \left( \frac{2Z + h}{2h} \right)^\nu + \rho_2 \]  

(3)

The strain-displacement relationships for a thin shell [14].

\[ \varepsilon_{11} = \frac{1}{A_1(1 + \frac{\alpha_3}{R_1})} \left( \frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_2} \frac{\partial \alpha_2}{\partial \alpha_1} + \frac{U_3}{A_3} \frac{\partial \alpha_3}{\partial \alpha_1} \right) \]  

(4)

\[ \varepsilon_{22} = \frac{1}{A_2(1 + \frac{\alpha_3}{R_2})} \left( \frac{\partial U_2}{\partial \alpha_2} + \frac{U_3}{A_3} \frac{\partial \alpha_3}{\partial \alpha_2} \right) \]  

(5)

\[ \varepsilon_{33} = \frac{\partial U_3}{\partial \alpha_3} \]  

(6)

\[ \varepsilon_{12} = \frac{A_3}{A_2(1 + \frac{\alpha_3}{R_2})} \frac{\partial}{\partial \alpha_2} \left( \frac{U_1}{A_1(1 + \frac{\alpha_3}{R_1})} \right) + \frac{1}{A_2(1 + \frac{\alpha_3}{R_2})} \frac{\partial U_3}{\partial \alpha_1} \]  

(7)

\[ \varepsilon_{13} = \frac{A_3}{A_1(1 + \frac{\alpha_3}{R_1})} \frac{\partial}{\partial \alpha_3} \left( \frac{U_1}{A_1(1 + \frac{\alpha_3}{R_1})} \right) + \frac{1}{A_1(1 + \frac{\alpha_3}{R_1})} \frac{\partial U_3}{\partial \alpha_1} \]  

(8)

\[ \varepsilon_{23} = \frac{A_3}{A_2(1 + \frac{\alpha_3}{R_2})} \frac{\partial}{\partial \alpha_3} \left( \frac{U_2}{A_2(1 + \frac{\alpha_3}{R_2})} \right) + \frac{1}{A_2(1 + \frac{\alpha_3}{R_2})} \frac{\partial U_3}{\partial \alpha_2} \]  

(9)

\[ A_1 = \left[ \frac{\partial \sigma}{\partial \alpha_1} \right] \, , \, A_2 = \left[ \frac{\partial \sigma}{\partial \alpha_2} \right] \]  

(10)

where \( A_1 \) and \( A_2 \) are the fundamental form parameters or Lame parameters, \( U_1 \), \( U_2 \) and \( U_3 \) are the displacement at any point \((\alpha_1, \alpha_2, \alpha_3)\), \( R_1 \) and \( R_2 \) are the radius of curvature related to \( \alpha_1 \), \( \alpha_2 \) and \( \alpha_3 \) respectively. The third- order theory of Reddy used in the present study is based on the following displacement field:

\[ \begin{align*}
U_1 &= u_1(\alpha_1, \alpha_2) + \alpha_3 \phi_1(\alpha_1, \alpha_2) + \alpha_2^2 \psi_1(\alpha_1, \alpha_2) + \alpha_1^2 \phi_2(\alpha_1, \alpha_2) + \alpha_1 \alpha_2 \phi_3(\alpha_1, \alpha_2) \\
U_2 &= u_2(\alpha_1, \alpha_2) + \alpha_3 \phi_1(\alpha_1, \alpha_2) + \alpha_2^2 \psi_2(\alpha_1, \alpha_2) + \alpha_1^2 \phi_2(\alpha_1, \alpha_2) + \alpha_1 \alpha_2 \phi_3(\alpha_1, \alpha_2) \\
U_3 &= u_3(\alpha_1, \alpha_2)
\end{align*} \]  

(11)

These equations can be reduced by satisfying the stress-free conditions on the top and bottom faces of the laminates, which are equivalent to \( \varepsilon_{13} = \varepsilon_{23} = 0 \) at \( Z = \pm \frac{h}{2} \). Thus,

\[ \begin{align*}
U_1 &= u_1(\alpha_1, \alpha_2) + \alpha_3 \phi_1(\alpha_1, \alpha_2) - C_1 \alpha_2 \left( \frac{u_1}{R_1} + \frac{\partial u_2}{\partial \alpha_1} \right) \\
U_2 &= u_2(\alpha_1, \alpha_2) + \alpha_3 \phi_1(\alpha_1, \alpha_2) - C_1 \alpha_2 \left( \frac{u_2}{R_2} + \frac{\partial u_3}{\partial \alpha_1} \right) \\
U_3 &= u_3(\alpha_1, \alpha_2)
\end{align*} \]  

(12)

Where \( C_1 = \frac{4}{3h^2} \). Substituting Eq. (12) into nonlinear strain-displacement relation (4) - (9), it can be obtained for the third-order theory of Reddy

\[ \begin{pmatrix} e_{11} \\ e_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} k_{11}^r \\ k_{22}^r \end{pmatrix} + \alpha_2^2 \begin{pmatrix} k_{11}^r \\ k_{22}^r \end{pmatrix} \]  

(13)

\[ \begin{pmatrix} e_{13} \\ e_{23} \end{pmatrix} = \begin{pmatrix} \gamma_{13}^r \\ \gamma_{23}^r \end{pmatrix} + \alpha_3 \begin{pmatrix} \gamma_{13}^r \\ \gamma_{23}^r \end{pmatrix} + \alpha_2^2 \begin{pmatrix} \gamma_{13}^r \\ \gamma_{23}^r \end{pmatrix} \]  

(14)
where

\[
\begin{align*}
\epsilon_{11} &= \begin{bmatrix} 0 \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} 1 & u_2 & \frac{\partial u_1}{\partial x_1} + \frac{u_3}{R_1} \\ -u_2 & 0 & \frac{\partial u_2}{\partial x_1} + \frac{u_1}{R_1} \\ -\frac{u_2}{R_1} & -\frac{u_2}{R_1} & 0 \end{bmatrix}, \\
\epsilon_{22} &= \begin{bmatrix} 0 \\ \epsilon_{12} \\ \epsilon_{22} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} + \frac{u_3}{R_1} \\ -\frac{\partial u_1}{\partial x_1} & 0 & \frac{\partial u_2}{\partial x_1} + \frac{u_1}{R_1} \\ -\frac{\partial u_2}{\partial x_1} & -\frac{\partial u_2}{\partial x_1} & 0 \end{bmatrix}, \\
\epsilon_{12} &= \begin{bmatrix} 0 \\ \epsilon_{12} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_1} + \frac{u_3}{R_1} \\ -\frac{\partial u_2}{\partial x_1} & 0 & \frac{\partial u_1}{\partial x_1} + \frac{u_1}{R_1} \\ -\frac{\partial u_1}{\partial x_1} & -\frac{\partial u_1}{\partial x_1} & 0 \end{bmatrix}.
\end{align*}
\]

(15)

\[
\begin{align*}
k_{11} &= \begin{bmatrix} 0 \\ k_{22} \\ k_{12} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} + \frac{u_3}{R_1} \\ -\frac{\partial u_1}{\partial x_1} & 0 & \frac{\partial u_2}{\partial x_1} + \frac{u_1}{R_1} \\ -\frac{\partial u_2}{\partial x_1} & -\frac{\partial u_2}{\partial x_1} & 0 \end{bmatrix}, \\
k_{22} &= \begin{bmatrix} 0 \\ k_{22} \\ k_{22} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} + \frac{u_3}{R_1} \\ -\frac{\partial u_1}{\partial x_1} & 0 & \frac{\partial u_2}{\partial x_1} + \frac{u_1}{R_1} \\ -\frac{\partial u_2}{\partial x_1} & -\frac{\partial u_2}{\partial x_1} & 0 \end{bmatrix}, \\
k_{12} &= \begin{bmatrix} 0 \\ k_{12} \\ k_{12} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} + \frac{u_3}{R_1} \\ -\frac{\partial u_1}{\partial x_1} & 0 & \frac{\partial u_2}{\partial x_1} + \frac{u_1}{R_1} \\ -\frac{\partial u_2}{\partial x_1} & -\frac{\partial u_2}{\partial x_1} & 0 \end{bmatrix}.
\end{align*}
\]

(16)

\[
\begin{align*}
\{\gamma\}_{13} &= \left\{ \phi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1}, \phi_2 - \frac{u_2}{R_2} + \frac{1}{A_2} \frac{\partial u_2}{\partial x_1}, \phi_3 - \frac{u_3}{R_1} + \frac{1}{A_1} \frac{\partial u_3}{\partial x_1} \right\}, \\
\{\gamma\}_{23} &= \left\{ \phi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1}, \phi_2 - \frac{u_2}{R_2} + \frac{1}{A_2} \frac{\partial u_2}{\partial x_1}, \phi_3 - \frac{u_3}{R_1} + \frac{1}{A_1} \frac{\partial u_3}{\partial x_1} \right\}.
\end{align*}
\]

(18)

\[
\begin{align*}
\gamma_{13} &= 3C_1 \begin{bmatrix} \chi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} \\ -\frac{u_2}{R_2} + \frac{1}{A_2} \frac{\partial u_2}{\partial x_1} \end{bmatrix}, \\
\gamma_{23} &= 3C_1 \begin{bmatrix} \chi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} \\ -\frac{u_2}{R_2} + \frac{1}{A_2} \frac{\partial u_2}{\partial x_1} \end{bmatrix}.
\end{align*}
\]

(19)

\[
\begin{align*}
\gamma_{11} &= \begin{bmatrix} \chi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} \\ \chi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} \\ \chi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} \end{bmatrix}, \\
\gamma_{22} &= \begin{bmatrix} \chi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} \\ \chi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} \\ \chi_1 - \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} \end{bmatrix}.
\end{align*}
\]

(20)

Where \((\epsilon^n, \gamma^n)\) are the membranes strains and \((k_k, k_k', \gamma_1, \gamma_1')\) are the bending strains, known as the curvatures.

2- FORMULATION

For a thin cylindrical shell, the stress -strain relationship are defined as

\[
\begin{align*}
\sigma_{11} &= \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{54} & 0 \\ 0 & 0 & 0 & 0 & Q_{56} \end{bmatrix}, \\
\sigma_{22} &= \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{54} & 0 \\ 0 & 0 & 0 & 0 & Q_{56} \end{bmatrix}, \\
\sigma_{12} &= \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{54} & 0 \\ 0 & 0 & 0 & 0 & Q_{56} \end{bmatrix}, \\
\sigma_{13} &= \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{54} & 0 \\ 0 & 0 & 0 & 0 & Q_{56} \end{bmatrix}, \\
\sigma_{12} &= \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{54} & 0 \\ 0 & 0 & 0 & 0 & Q_{56} \end{bmatrix}.
\end{align*}
\]

(21)
For an isotropic cylindrical shell the reduced stiffness $Q_{ij}$ ($i, j = 1, 2$ and $6$) are defined as

$$Q_{11} = Q_{22} = \frac{E}{1 - \nu^2}, Q_{12} = \frac{\nu E}{1 - \nu^2}$$

$$Q_{44} = Q_{55} = Q_{66} = \frac{E}{2(1 + \nu)}$$

where $E$ is the Young's modulus and $\nu$ is Poisson's ratio. Defining

$$[A_{ij}, B_{ij}, D_{ij}, E_{ij}, G_{ij}, H_{ij}] = \int_{\Omega}^{H} Q_{ij} \left[ 1, \alpha_3^2, \alpha_3^4, \alpha_3^6, \alpha_3^8, \alpha_3^10 \right] \, d\alpha$$

where $Q_{ij}$ are functions of $z$ for functionally gradient materials. Here $A_{ij}$ denote the extensional stiffness, $D_{ij}$ the bending stiffness, $B_{ij}$ the bending-extensional coupling stiffness and $E_{ij}, F_{ij}, G_{ij}, H_{ij}$ are the extensional, bending, coupling, and higher-order stiffness. For a thin cylindrical shell the force and moment results are defined as

$$[N_{11}, N_{22}, N_{12}] = \left[ \frac{h}{2} \sigma_{22}, \frac{h}{2} \sigma_{22}, \alpha_3 \sigma_{22} \right] \, d\alpha_3$$

$$[M_{11}, M_{22}, M_{12}] = \left[ \frac{h}{2} \sigma_{22}, \frac{h}{2} \sigma_{22}, \alpha_3^2 \sigma_{22} \right] \, d\alpha_3$$

$$[P_{11}, P_{22}, P_{12}] = \left[ \frac{h}{2} \sigma_{22}, \frac{h}{2} \sigma_{22}, \alpha_3^3 \sigma_{22} \right] \, d\alpha_3$$

$$[Q_{13}, Q_{23}] = \left[ \frac{h}{2} \sigma_{22}, \frac{h}{2} \sigma_{22}, \alpha_3^4 \sigma_{22} \right] \, d\alpha_3$$

$$[R_{13}, R_{23}] = \left[ \frac{h}{2} \sigma_{22}, \frac{h}{2} \sigma_{22}, \alpha_3^5 \sigma_{22} \right] \, d\alpha_3$$

3- The equations of motion for vibration of a generic shell

The equations of motion for vibration of a generic shell can be derived by using Hamilton's principle which is described by

$$\delta \int_{t_1}^{t_2} (\Pi - K) \, dt = 0 \quad , \quad \Pi = U - V$$

Where $K, \Pi, U$ and $V$ are the total kinetic, potential, strain and loading energies, $t_1$ and $t_2$ are arbitrary time. The kinetic, strain and loading energies of a cylindrical shell can be written as:

$$K = \frac{1}{2} \int \int \rho (U_1^2 + U_2^2 + U_3^2) \, dV$$

$$U = \int \left( \sigma_{11} \varepsilon_{11} + \sigma_{22} \varepsilon_{22} + \sigma_{12} \varepsilon_{12} + \sigma_{13} \varepsilon_{13} + \sigma_{23} \varepsilon_{23} \right) \, dV$$

$$V = \int \int \left( q_1 \partial U_1 + q_2 \partial U_2 + q_3 \partial U_3 \right) A_i \, d\alpha_i \, d\alpha_3$$

The infinitesimal volume is given by

$$dV = A_i A_3 \, d\alpha_i \, d\alpha_3$$

4- Equations of motion for vibration of cylindrical shell

the equations of motions for vibration of cylindrical shell with the third-order theory of Reddy are converted to
The displacement fields for a FG cylindrical shell and the displacement fields which satisfy these boundary conditions can be written as

\[ u_1 = \hat{A} \frac{\partial \phi(x)}{\partial x} \cos(n\theta) \cos(\omega t) \]
\[ u_2 = \hat{B} \phi(x) \sin(n\theta) \cos(\omega t) \]
\[ u_3 = \hat{C} \phi(x) \cos(n\theta) \cos(\omega t) \]
\[ \phi_1 = \hat{D} \frac{\partial \phi(x)}{\partial x} \cos(n\theta) \cos(\omega t) \]
\[ \phi_2 = \hat{E} \phi(x) \sin(n\theta) \cos(\omega t) \]

where, \( \hat{A}, \hat{B}, \hat{C}, \hat{D}, \) and \( \hat{E} \) are the constants denoting the amplitudes of the vibrations in the \( x, \theta \) and \( z \) directions, \( \phi_1 \) and \( \phi_2 \) are the displacement fields for higher order deformation theories for a cylindrical shell, \( \phi(x) \) is the axial function that satisfies the geometric boundary conditions. The axial function \( \phi(x) \) is chosen as the beam function as

\[ \phi(x) = \gamma_1 \frac{\lambda x}{L} \cos(\frac{\lambda x}{L}) + \gamma_2 \frac{\lambda x}{L} \cos(\frac{\lambda x}{L}) - \gamma_3 \frac{\lambda x}{L} \sin(\lambda x) + \gamma_4 \sin(\frac{\lambda x}{L}) \]

Substituting Eq. (38) into Eqs. (33) - (37) for third order theory we can be expressed

\[ \det (C_{ij} - M_{ij} \omega^2) = 0 \]

Expanding this determinant, a polynomial in even powers of \( \omega \) is obtained

\[ \beta_1 \omega^{10} + \beta_2 \omega^8 + \beta_3 \omega^6 + \beta_4 \omega^4 + \beta_5 \omega^2 + \beta_6 \omega^0 = 0 \]

where \( \beta_i (i = 0, 1, 2, 3, 4, 5) \) are some constants. Eq. (41) is solved five positive and five negative roots are obtained. The five positive roots obtained are the natural angular frequencies of the cylindrical shell based third-order theory. The smallest of the five roots is the natural angular frequency studied in the present study.
5- RESULTS AND DISCUSSION

Studies are presented for vibration of FG cylindrical shell. The boundary conditions, free-simply support (F-SS) is considered in the study. Natural frequencies of the FG cylindrical shell for this boundary conditions is computed and plotted in Fig. 1. For this boundary conditions the frequency first decreases and then increases as the circumferential wave number n increases.

![Figure 1: Natural frequencies FGM cylindrical shell associated with F-SS boundary conditions.](m=1, h/R=0.002, L/R=20)

6- CONCLUSIONS

The study is carried out using third order shear deformation shell theory. The analysis is carried out using Hamilton’s principle. Studies are carried out for cylindrical shells with free-simply support (F-SS) boundary conditions. The study showed that in this boundary conditions the frequency first decreases and then increases as the circumferential wave number n increases with free-free boundary conditions.

REFERENCES