Vibration FGM Cylindrical Shell with Effects Free Boundary Conditions

Mohammad Setareh¹ and Mohammad Reza Isvandzibaei²

¹,²Department of Mechanical Engineering, Andimeshk Branch, Islamic Azad University, Andimeshk, Iran

ABSTRACT

Study of the vibration cylindrical shells is very important. Material properties are graded in the thickness direction of the shell according to volume fraction power law distribution. The objective is to study the natural frequencies, the influence of constituent volume fractions and the effects of boundary conditions on the natural frequencies of the FGM cylindrical shell. The study is carried out using third order shear deformation shell theory. The governing equations of motion of FGM cylindrical shells are derived based on shear deformation theory. Results are presented on the frequency characteristics and the effects of free boundary conditions.

KEY WORDS: Vibration, FGM, Cylindrical shell, Free Conditions.

INTRODUCTION

Understanding of vibration behavior of cylindrical shells is an important aspect for the successful applications of cylindrical shells. Researches on free vibrations of cylindrical shells have been carried out extensively [1-5]. Recently, the present authors presented studies on the influence of boundary conditions on the frequencies of a multi–layered cylindrical shell [6]. In all the above works, different thin shell theories based on Love–hypothesis were used.

Vibration of cylindrical shells with ring support is considered by Loy and Lam [7]. The concept of functionally graded materials (FGMs) was first introduced in 1984 by a group of materials scientists in Japan [8-9] as a means of preparing thermal barrier materials. Since then, FGMs have attracted much interest as heat-shielding materials.

FGMs are made by combining different materials using power metallurgy methods [10]. They possess variations in constituent volume fractions that lead to continuous change in the composition, microstructure, porosity, etc., resulting in gradients in the mechanical and thermal properties [11-12]. Vibration study of FGM shell structures is important. However, study of the vibration of FGM shells with ring supports is limited.

The FGMs considered are composed of stainless steel and nickel where the volume fractions follow a power-law distribution. The study is carried out based on third order shear deformation shell theory. Studies are carried out for cylindrical shells with free (F-F) boundary conditions. Results presented include the frequency characteristics of cylindrical shells, and the influence of boundary conditions. The present analysis is validated by comparing results with others in the literature.

1- FUNCTIONALLY GRADED MATERIAL

For the cylindrical shell made of FGM the material properties such as the modulus of elasticity $E$, Poisson ratio $\nu$ and the mass density $\rho$ are assumed to be functions of the volume fraction of the constituent materials when the coordinate axis across the shell thickness is denoted by $z$ and measured from the shell’s middle plane. The functional relationships between $E$, $\nu$ and $\rho$ with $z$ for a stainless steel and nickel FGM shell are assumed as [13].

$$E = (E_s - E)\left(\frac{2Z + h}{2h}\right)^{n} + E_s$$

$$\nu = (\nu_s - \nu)\left(\frac{2Z + h}{2h}\right)^{n} + \nu_s$$

*Corresponding Author: Mohammad Reza Isvandzibaei is Ph.D. Student in Mechanical Engineering. Email: esvandzebaei@yahoo.com
\begin{equation}
\rho = (\rho_1 - \rho_2) \left( \frac{2Z + h}{2h} \right) + \rho_2
\end{equation}

The strain-displacement relationships for a thin shell [14].

\begin{equation}
\varepsilon_{11} = \frac{1}{A_1(1 + \frac{a_3}{R_1})} \begin{bmatrix} U_1 & U_2 & \frac{\partial U_3}{\partial a_1} \end{bmatrix}
\end{equation}

\begin{equation}
\varepsilon_{22} = \frac{1}{A_2(1 + \frac{a_3}{R_2})} \begin{bmatrix} U_2 & U_3 & \frac{\partial U_3}{\partial a_2} \end{bmatrix}
\end{equation}

\begin{equation}
\varepsilon_{33} = \frac{\partial U_3}{\partial a_3}
\end{equation}

\begin{equation}
\varepsilon_{12} = \frac{A_1(1 + \frac{a_3}{R_1}) \frac{\partial}{\partial a_1} (\frac{U_1}{A_1(1 + \frac{a_3}{R_1})} + \frac{A_2(1 + \frac{a_3}{R_2}) \frac{\partial}{\partial a_3} (\frac{U_2}{A_2(1 + \frac{a_3}{R_2})})}{A_2(1 + \frac{a_3}{R_2})} + \frac{1}{A_1(1 + \frac{a_3}{R_1})} \frac{\partial U_3}{\partial a_1} + \frac{1}{A_2(1 + \frac{a_3}{R_2})} \frac{\partial U_3}{\partial a_2}
\end{equation}

\begin{equation}
A_1 = \begin{bmatrix} \frac{\partial \sigma}{\partial a_1} \end{bmatrix} , \quad A_2 = \begin{bmatrix} \frac{\partial \sigma}{\partial a_2} \end{bmatrix}
\end{equation}

where $A_1$ and $A_2$ are the fundamental form parameters or Lame parameters, $U_1$, $U_2$ and $U_3$ are the displacement at any point ($a_1, a_2, a_3$), $R_1$ and $R_2$ are the radius of curvature related to $a_1, a_2$ and $a_3$ respectively. The third-order theory of Reddy used in the present study is based on the following displacement field:

\begin{equation}
\begin{bmatrix} U_1 = u_1(a_1, a_2) + \alpha_1 \dot{\phi}_1(a_1, a_2) + \alpha_1^2 \ddot{\phi}_1(a_1, a_2) + \alpha_1^3 \dddot{\phi}_1(a_1, a_2) \\
U_2 = u_2(a_1, a_2) + \alpha_1 \dot{\phi}_2(a_1, a_2) + \alpha_1^2 \ddot{\phi}_2(a_1, a_2) + \alpha_1^3 \dddot{\phi}_2(a_1, a_2) \\
U_3 = u_3(a_1, a_2)
\end{bmatrix}
\end{equation}

These equations can be reduced by satisfying the stress-free conditions on the top and bottom faces of the laminates, which are equivalent to $\varepsilon_{13} = \varepsilon_{23} = 0$ at $Z = \pm \frac{h}{2}$. Thus,

\begin{equation}
\begin{bmatrix} U_1 = u_1(a_1, a_2) + \alpha_1 \dot{\phi}_1(a_1, a_2) - C_1 a_1 (-\frac{u_1}{R_1} + \frac{\sigma_{13}}{A_0 a_1}) \\
U_2 = u_2(a_1, a_2) + \alpha_1 \dot{\phi}_2(a_1, a_2) - C_1 a_2 (-\frac{u_2}{R_2} + \frac{\sigma_{13}}{A_0 a_2}) \\
U_3 = u_3(a_1, a_2)
\end{bmatrix}
\end{equation}

Where $C_1 = \frac{4}{3h^2}$. Substituting Eq. (12) into nonlinear strain-displacement relation (4) - (9), it can be obtained for the third-order theory of Reddy

\begin{equation}
\begin{bmatrix} \varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{12}
\end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\
\varepsilon_{22} + \alpha_3 \begin{bmatrix} k_{11} \\
k_{22} \\
k_{12}
\end{bmatrix} \\
\varepsilon_{12}
\end{bmatrix}
\end{equation}

\begin{equation}
\begin{bmatrix} \varepsilon_{13} \\
\varepsilon_{23}
\end{bmatrix} = \begin{bmatrix} \gamma_1 \\
\gamma_2
\end{bmatrix} + \alpha_3 \begin{bmatrix} \gamma_{13} \\
\gamma_{23}
\end{bmatrix}
\end{equation}

where

\begin{equation}
\begin{bmatrix} \varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{12}
\end{bmatrix} = \begin{bmatrix} \frac{1}{A_1} \sigma_{11} + u_1 \frac{\partial U_1}{\partial a_1} + u_2 \frac{\partial U_1}{\partial a_2} \\
\frac{1}{A_2} \sigma_{22} + u_1 \frac{\partial U_2}{\partial a_1} + u_2 \frac{\partial U_2}{\partial a_2} \\
\frac{1}{A_1} \sigma_{12} + u_1 \frac{\partial U_3}{\partial a_1} + u_2 \frac{\partial U_3}{\partial a_2}
\end{bmatrix}
\end{equation}
the curvatures.

**FORMULATION**

Consider a cylindrical shell as shown in Fig. 2, where $R$ is the radius, $L$ the length and $h$ the thickness of the shell. The reference surface is chosen to be the middle surface of the cylindrical shell where an orthogonal coordinate system $x, \theta, z$ is fixed. The displacements of the shell with reference this coordinate system are denoted by $U_1, U_2$ and $U_3$ in the $x, \theta$ and $z$ directions, respectively.

![Fig 1: Geometry of FGM cylindrical shell](image)

For a thin cylindrical shell, the stress-strain relationship are defined as

$$
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix} = 
\begin{bmatrix}
\frac{Q_{11}}{R} & \frac{Q_{12}}{R} & 0 & 0 & 0 \\
\frac{Q_{12}}{R} & \frac{Q_{22}}{R} & 0 & 0 & 0 \\
0 & 0 & Q_{33} & 0 & 0 \\
0 & 0 & 0 & Q_{44} & 0 \\
0 & 0 & 0 & 0 & Q_{66}
\end{bmatrix} \begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{12} \\
\epsilon_{23} \\
\epsilon_{13}
\end{bmatrix}
$$

(20)
For a isotropic cylindrical shell the reduced stiffness \( Q_{ij} \) \((i, j=1, 2 \text{ and } 6)\) are defined as
\[
Q_{11} = Q_{22} = \frac{E}{1-\nu^2} \cdot Q_{12} = \frac{\nu E}{1-\nu^2}
\]
(22)
\[
Q_{44} = Q_{25} = Q_{66} = \frac{E}{2(1+\nu)}
\]
(23)
where \( E \) is the Young's modulus and \( \nu \) is Poisson's ratio. Defining
\[
\begin{align*}
A_{ij} & = \int_{0}^{\frac{2}{3}} Q_{ij} \sigma_1 \sigma_2 \alpha_3 d\alpha_3 \\
A_{ij} & = \int_{0}^{\frac{2}{3}} Q_{ij} \sigma_2 \sigma_2 \alpha_2^3 d\alpha_3 \\
A_{ij} & = \int_{0}^{\frac{2}{3}} Q_{ij} \sigma_1 \sigma_1 \alpha_1^3 d\alpha_3
\end{align*}
\]
(24)
where \( Q_{ij} \) are functions of \( z \) for functionally gradient materials. Here \( A_{ij} \) denote the extensional stiffness, \( D_{ij} \) the bending stiffness, \( B_{ij} \) the bending-extensional coupling stiffness and \( E_{ij}, F_{ij}, G_{ij}, H_{ij} \) are the extensional, bending, coupling, and higher-order stiffness. For a thin cylindrical shell the force and moment results are defined as
\[
\begin{align*}
N_{11} & = \int_{0}^{\frac{2}{3}} \sigma_1 \sigma_2 \sigma_2 d\alpha_3 \\
N_{22} & = \int_{0}^{\frac{2}{3}} \sigma_2 \sigma_2 \sigma_2 d\alpha_3 \\
N_{12} & = \int_{0}^{\frac{2}{3}} \sigma_1 \sigma_1 \sigma_2 d\alpha_3
\end{align*}
\]
(25)
\[
\begin{align*}
P_{11} & = \int_{0}^{\frac{2}{3}} \sigma_1 \sigma_1 \sigma_3 d\alpha_3 \\
P_{22} & = \int_{0}^{\frac{2}{3}} \sigma_2 \sigma_2 \sigma_3 d\alpha_3 \\
P_{12} & = \int_{0}^{\frac{2}{3}} \sigma_1 \sigma_1 \sigma_3 d\alpha_3
\end{align*}
\]
(26)
\[
\begin{align*}
Q_{13} & = \int_{0}^{\frac{2}{3}} \sigma_3 \sigma_1 d\alpha_3 \\
Q_{23} & = \int_{0}^{\frac{2}{3}} \sigma_3 \sigma_2 d\alpha_3 \\
Q_{33} & = \int_{0}^{\frac{2}{3}} \sigma_3 \sigma_3 d\alpha_3
\end{align*}
\]
(27)

3- The Equations of Motion for Vibration of a Generic Shell

The equations of motion for vibration of a generic shell can be derived by using Hamilton's principle which is described by
\[
\delta \int_{t_1}^{t_2} (\Pi - K) dt = 0 \quad , \quad \Pi = U - V
\]
(28)
Where \( K, \Pi, U \) and \( V \) are the total kinetic, potential, strain and loading energies, \( t_1 \) and \( t_2 \) are arbitrary time. The kinetic, strain and loading energies of a cylindrical shell can be written as:
\[
K = \frac{1}{2} \int \int \rho(U_x^2 + U_y^2 + U_z^2) dV
\]
(29)
\[
U = \int \int_{\alpha_1, \alpha_2} \left( \sigma_1 \epsilon_{11} + \sigma_2 \epsilon_{22} + \sigma_3 \epsilon_{33} + \sigma_{12} \epsilon_{12} + \sigma_{13} \epsilon_{13} + \sigma_{23} \epsilon_{23} \right) dV
\]
(30)
\[
V = \int \int \left( q_1 \tilde{U}_1 + q_2 \tilde{U}_2 + q_3 \tilde{U}_3 \right) A_{ij} d\alpha_i d\alpha_j
\]
(31)
The infinitesimal volume is given by
\[
dV = A_{ij} d\alpha_i d\alpha_j
\]
(32)
with the use of Eqs. (11)-(20) and substituting into Eq. (28), we get the equations of motions of a generic shell.
\[
\begin{align*}
\frac{\partial N(A)}{\partial t} + N(A) + \left( \frac{\partial N(A)}{\partial x} \right) \frac{\partial A}{\partial A} + \frac{\partial A}{\partial A} - \frac{\partial P(CA)}{\partial x} & = 0 \\
\frac{\partial A}{\partial x} \frac{\partial A}{\partial x} & = 0
\end{align*}
\]
(33)
\[
\begin{align*}
\frac{\partial N_{11}}{\partial t} + \frac{\partial N_{12}}{\partial x} &= I_{b1} \ddot{u}_1 + (I_1 - C_1 I_3) \ddot{\theta}_1 - C_1 \left( \frac{\partial^2 \ddot{u}_1}{\partial x^2} \right), \\
\frac{\partial N_{12}}{\partial t} + \frac{\partial N_{13}}{\partial x} &= I_{b2} \ddot{u}_2 + (I_2 - C_1 I_3) \ddot{\theta}_1 + C_1 \left( \frac{\partial^2 \ddot{u}_2}{\partial x^2} \right) + C_1 \left( \frac{\partial^2 \ddot{u}_1}{\partial x^2} \right),
\end{align*}
\]
(34)

For Eqs. (33) – (37) are defining as

\[
I_i = \int_{-\frac{t}{2}}^{\frac{t}{2}} \rho \alpha \, d \alpha_i
\]
(38)

4- Equations of motion for vibration of cylindrical shell

The curvilinear coordinates and fundamental from parameters for a cylindrical shell are:

\[
R_2 = a = 0, A_2 = a, A_3 = \alpha, \alpha_3 = 0, \alpha_1 = x
\]
(39)

Substituting relationship (39) into Eqs. (33)-(37) the equations of motions for vibration of cylindrical shell with the third-order theory of Reddy are converted to

\[
\begin{align*}
\frac{a}{R_2} \frac{\partial \ddot{N}_{11}}{\partial x} + \frac{\partial \ddot{N}_{12}}{\partial x} &= I_{b1} \ddot{u}_1 + (I_1 - C_1 I_3) \ddot{\theta}_1 - C_1 \left( \frac{\partial^2 \ddot{u}_1}{\partial x^2} \right), \\
\frac{a}{R_2} \frac{\partial \ddot{N}_{12}}{\partial x} + \frac{\partial \ddot{N}_{13}}{\partial x} &= I_{b2} \ddot{u}_2 + (I_2 - C_1 I_3) \ddot{\theta}_1 + C_1 \left( \frac{\partial^2 \ddot{u}_2}{\partial x^2} \right) + C_1 \left( \frac{\partial^2 \ddot{u}_1}{\partial x^2} \right),
\end{align*}
\]
(40)

\[
\begin{align*}
\frac{C_1}{a} \frac{\partial^2 \ddot{u}_1}{\partial x^2} + \frac{C_1}{a} \frac{\partial^2 \ddot{u}_2}{\partial x^2} &= (I_1 - C_1 I_3) \ddot{\theta}_1 - C_1 \left( \frac{\partial^2 \ddot{u}_1}{\partial x^2} \right) + \frac{C_1}{a} \left( \frac{\partial^2 \ddot{u}_1}{\partial x^2} \right),
\end{align*}
\]
(41)
To validate the present analysis, results for cylindrical shells are compared with Loy and Lam [15] and with M.R.Isvandizibaei [16]. The comparisons show that the present results agreed well with those in the literature.

5 RESULTS AND DISCUSSION

The displacement fields for a FG cylindrical shell and the displacement fields which satisfy these boundary conditions can be written as

\[ u_i = A \frac{\partial^2 \phi(x)}{\partial x^2} \cos(n\theta) \cos(\omega t) \]
\[ u_2 = B \phi(x) \sin(n\theta) \cos(\omega t) \]
\[ u_3 = C \phi(x) \cos(n\theta) \cos(\omega t) \]
\[ v_1 = D \frac{\partial \phi(x)}{\partial x} \cos(n\theta) \cos(\omega t) \]
\[ v_2 = E \phi(x) \sin(n\theta) \cos(\omega t) \]

where, \(A, B, C, D, E\) and \(\phi\) are the constants denoting the amplitudes of the vibrations in the \(x, \theta, \) and \(z\) directions, \(\phi_1\) and \(\phi_2\) are the displacement fields for higher order deformation theories for a cylindrical shell, \(\phi(x)\) is the axial function that satisfies the geometric boundary conditions. The axial function \(\phi(x)\) is chosen as the beam function as

\[ \phi(x) = \gamma_1 \cosh \frac{\lambda x}{L} + \gamma_2 \cos \frac{\lambda x}{L} - \zeta_1 \sinh \frac{\lambda x}{L} + \zeta_2 \sin \frac{\lambda x}{L} \]

The geometric boundary conditions for free boundary conditions can be expressed mathematically in terms of \(\phi(x)\) as:

\[ \phi''(0) = \phi''(L) = 0 \]

Substituting Eq. (45) into Eqs. (40) - (44) for third order theory we can be expressed

\[ \det(C_{ij} - M_{ij} \omega^2) = 0 \]

Expanding this determinant, a polynomial in even powers of \(\omega\) is obtained

\[ \beta_1 \omega^{10} + \beta_2 \omega^{12} + \beta_3 \omega^{14} + \beta_4 \omega^{16} + \beta_5 \omega^{18} + \beta_6 = 0 \]

where \(\beta_i(i = 0, 1, 2, 3, 4, 5)\) are some constants. Eq. (49) is solved five positive and five negative roots are obtained. The five positive roots obtained are the natural angular frequencies of the cylindrical shell based third-order theory. The smallest of the five roots is the natural angular frequency studied in the present study.

5 RESULTS AND DISCUSSION
Table 1: Comparison of natural frequency (Hz) for a clamped isotropic cylindrical shell.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2043.8</td>
<td>2043.6</td>
<td>2045.1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5635.4</td>
<td>5635.2</td>
<td>5624.6</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>8932.5</td>
<td>8932.1</td>
<td>8821.5</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>11407.5</td>
<td>11407.2</td>
<td>11437</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>13253.2</td>
<td>13252.8</td>
<td>13197.5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>14790.0</td>
<td>14789.8</td>
<td>14790.6</td>
</tr>
</tbody>
</table>

In this paper, studies are presented for a FGM cylindrical shell with free-free boundary conditions. Table 2 shows the variation of the natural frequency with the circumferential wave number n for a functional graded cylindrical shell. The frequencies for the free-free boundary conditions increased with the circumferential wave number.

Table 2: The natural frequencies for a FGM cylindrical shell under (F-F) boundary conditions (m = 1, h / R=0.01, L / R=20).

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>$\omega$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0.376687</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.472224</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.496101</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>0.506079</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>0.513007</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>0.520445</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>0.530317</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>0.544065</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>0.562919</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>0.587923</td>
</tr>
</tbody>
</table>

Studies are presented for vibration of FG cylindrical shell. The boundary conditions, free-free (F-F) is considered in the study. Natural frequencies of the FG cylindrical shell for this boundary conditions is computed and plotted in Fig. 2. For this boundary conditions the frequency first decreases and then increases as the circumferential wave number n increases.

Figure 2: Natural frequencies FGM cylindrical shell associated with F-F boundary conditions. (m=1, h/R=0.002, L/R=20)

6- CONCLUSIONS

Material properties are graded in the thickness direction of the shell according to volume fraction power law distribution. The study is carried out using third order shear deformation shell theory. The analysis is carried out using Hamilton’s principle. Studies are carried out for cylindrical shells with free-free (F-F) boundary conditions. The study showed that in this boundary conditions the frequency first decreases and then increases as the circumferential wave number n increases with free-free boundary conditions.
REFERENCES


