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Construction of Serendipity Shape Functions by Geometrical Probability

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ABSTRACT

In this paper the rectangular finite element (FE) is considered as a serendipity family (O.Zienkiewicz, R.Taylor, 2000): linear (4 nodes), quadratic (8 nodes), cubic (12 nodes). A cubic FE is received with standard (12 parameters) and modified (13 parameters) are functions of such a form. The paper shows a way of correcting the deficiencies of shape functions forms FESF using a weighted average of various models.

KEYWORDS: Geometrical probability, finite element method, serendipity family, parametric interpolation.

1. INTRODUCTION

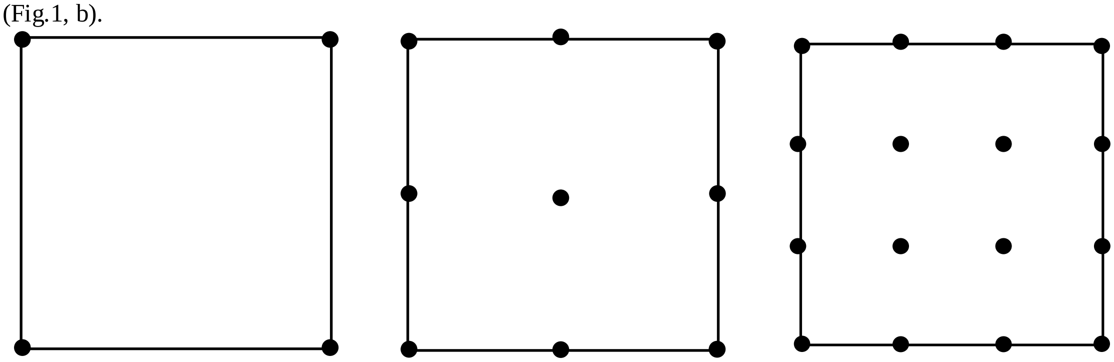
Finite elements of serendipity family (FESF) form are a useful class of discrete models. For the first time these elements were obtained in 1968 with the help of resourceful recruiting because they did not lend any formalization. FESF are the result of modification of FE Lagrangian families by eliminating internal nodes. The focus of this paper is on cubic FE serendipity family. In the standard model parameters, an interpolation polynomial coincides with the number of nodes on the boundary (12). There are several ways of solving the problem of constructing the standard functions of the form: inverse matrix method, systematic generation of shape functions (Taylor's procedure), and the probabilistic-geometrical method (PGM). The main advantage of PGM over other methods is that it allows discovering the hidden parameters in the interpolation procedure. For example, the cubic FE PGM provides a 13-parameter interpolation. The corresponding shape functions of here are given in explicit form. These results indicate that at FESF higher orders, there are many suitable bases. This means that the PGM has opened new opportunities for solving urgent problems of optimizing functions of such a form. The paper shows a way of correcting the deficiencies of functions forms FESF using a weighted average of various models. In this paper prove the cubic FE serendipity family; there is a 13-parametric interpolation polynomial that satisfies the conditions of Lagrange.

2. RESULTS AND DISCUSSION

In FEM initially applied only finite elements to the family of Lagrangian [1-3, 6, 7]. Options of Lagrangian form of family (Fig.1, a) are the natural generalization of the one-dimensional Lagrangian interpolation. These functions are obtained by multiplying the one-dimensional polynomials (separation of variables). Interpolation and the computational drawbacks of Lagrangian family are well known [1, 4-7].

If the FE families of Lagrangian exclude parasitic internal nodes, we get FE serendipity family (Fig.1, b).

(a)



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(b)

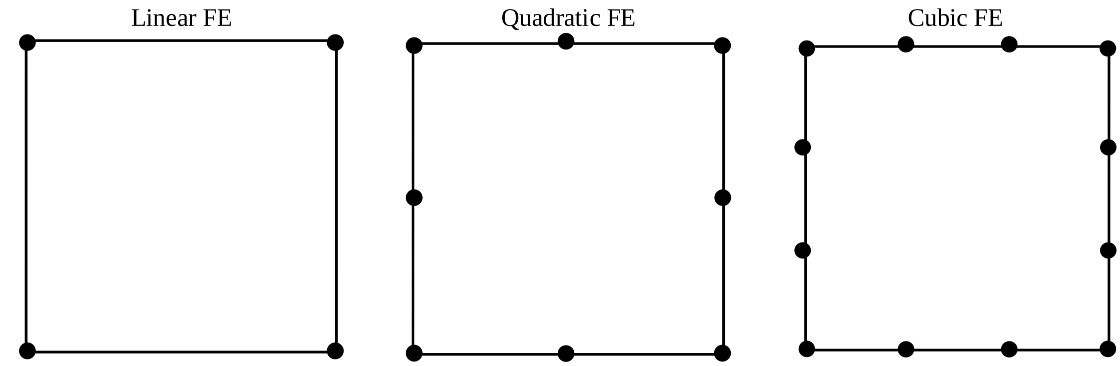

Fig.1 Square elements: (a) Lagrang family; (b) “serendipity” family

The key idea of PGM and the procedure for applying the method in problems of constructing shape functions of Lagrangian elements are described in [2, 3]. In this paper the effectiveness of geometric probability in FE serendipity family is considered. First, the quadratic FE serendipity family is considered. As usual [4-7] is a square 2x2 ($|\xi| \le 1, |\eta| \le 1$), 8 nodes are regularly located on its border (Fig.2, a).

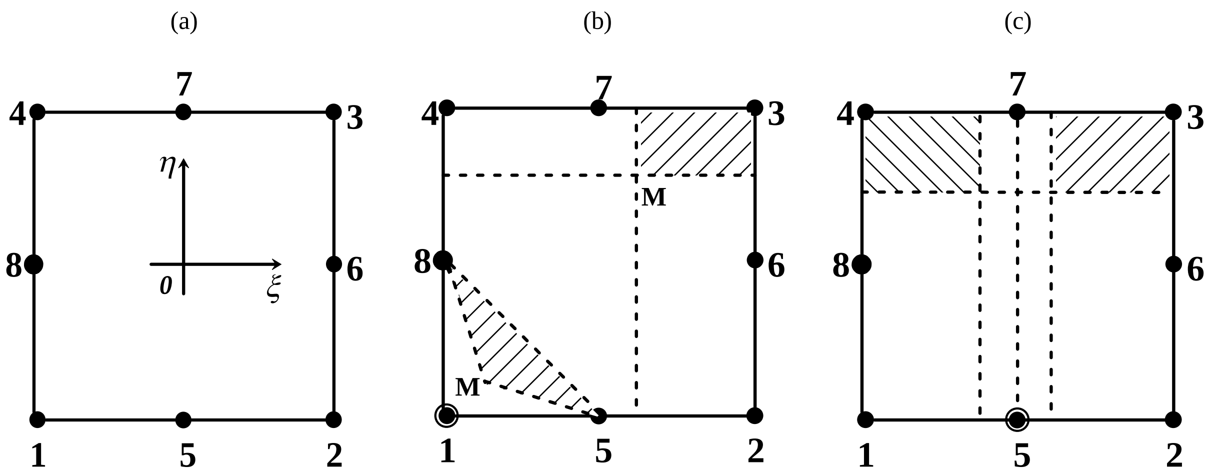

Fig.2 Quadratic FE: (a) general view; (b) composition of simple elements for $N_1(\xi,\eta)$; (c) composition of simple elements for $N_5(\xi,\eta)$

Baseline data contain 24 numbers: the coordinates of nodes (ξ_i,η_i) and the nodal values of the function $f_i,\ i=\overline{1,8}$. The challenge is to construct a polynomial interpolation

$$P(\xi,\eta)=\sum_{i=1}^{8} N_i(\xi,\eta)\cdot f_i \tag{1}$$

The function form must possess the following characteristics (conditions of Lagrange type):

$$N_i(\xi_k,\eta_k)=\delta_{ik} \tag{2}$$

where δ_{ik} - is the Kronecker delta, i is the number of functions and k is the number of nodes;

$$\sum_{i=1}^{8} N_i(\xi,\eta)=1 \tag{3}$$

Besides, function of the form $N_i(\xi,\eta)$ is to ensure C^o continuity at the boundary: if the node i belongs to a particular side of the square, then the function $N_i(\xi,\eta)$ along this part of changing the law of quadratic parabola (3 knots).

First, two well-known method of solution of the problem are briefly described.

2.1. Algebraic (matrix) method [4, 5].

Initially interpolation polynomial is written in a general terms (by means of Pascal's triangle):

$$P(\xi,\eta)=\alpha_1+\alpha_2\xi+\alpha_3\eta+\alpha_4\xi^2+\alpha_5\xi\eta+\alpha_6\eta^2+\alpha_7\xi^2\eta+\alpha_8\xi\eta^2, \tag{4}$$

uncertain function. α_i $\left(i=\overline{1,8}\right)$ where

Now using the input data , a system of linear algebraic equations 8x8 is represented:

$$\alpha_1+\alpha_2\xi_i+\alpha_3\eta_i+\alpha_4\xi_i^2+\alpha_5\xi_i\eta_i+\alpha_6\eta_i^2+\alpha_7\xi_i^2\eta_i+\alpha_8\xi_i\eta_i^2=f_i, \quad i=\overline{1,8} \tag{5}$$

The determinant of a matrix system (5) is not equal to zero, which guarantees the uniqueness of the solution. Solving system (5), we get only one set of coefficients α_i , which are expressed in terms of the 24 number $\{\xi_i,\eta_i,\ f_i\}$.

The substitution of α_i into (4) and rearrangement of terms on the polynomial f_i leads to a type (1). In doing so, we get the function form in an explicit form:

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)(\xi_i\xi+\eta_i\eta-1),\ i=\overline{1,4};\ \xi_i,\eta_i=\pm1 \tag{6}$$

$$N_i(\xi,\eta)=\frac{1}{2}\left(1-\xi^2\right)(1+\eta_i\eta),\quad i=5,7;\ \eta_i=\pm1 \tag{7}$$

$$N_i(\xi,\eta)=\frac{1}{2}\left(1-\eta^2\right)(1+\xi_i\xi),\quad i=6,8;\ \xi_i=\pm1 \tag{8}$$

2.2. Taylor procedure [1, 6, 7]. Initially the shape functions of nodes in the middle of the parties is designed: $i=5,6,7,8$. To this end, the corresponding Lagrange polynomial of second degree in one direction is multiplied by the linear function in other areas. This allows the parties to obtain the desired degree of the polynomial. This automatically is condition (2).

For an angular node this method does not give the desired result, as along as one side shape function changes linearly, so in the middle it gives 0.5 instead of zero. However, a linear combination of bilinear functions, and functions of the form secondary nodes give the desired result.

For example,

$$N_5(\xi,\eta)=\left(1-\xi^2\right)\cdot\frac{1}{2}(1-\eta)$$

$$N_8(\xi,\eta)=\left(1-\eta^2\right)\cdot\frac{1}{2}(1-\xi).$$

make a linear combination of bilinear function, $N_1(\xi,\eta)$ Now, to get

$$\overline{N_1}(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$$

and $N_8(\xi,\eta)$ and the forms functions: $N_5(\xi,\eta)$

$$N_1(\xi,\eta)=\overline{N_1}(\xi,\eta)-\frac{1}{2}N_5(\xi,\eta)-\frac{1}{2}N_8(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)-\frac{1}{4}\left(1-\xi^2\right)(1-\eta)-$$
$$-\frac{1}{4}\left(1-\eta^2\right)(1-\xi)=\frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1) \quad (9)$$

As it can be seen, the procedure of Taylor leads to the same result. This should be expected, since the determinant of a matrix system (5) is non zero.

2.3.Application of PGM [2, 3]. Note that the FEs of higher order can always be represented as compositions of simple FE. The number of simple elements is determined by the order of the polynomial of interpolation. In this case, each node is associated with two simple FE.

For example, in the design $N_1(\xi,\eta)$, two simple FEs are considered:1-2-3-4, and 1-5-8 (Fig.2, b). In each of these FEs a random point is throwing . In Fig.2, b, c the area of "successful" outcomes is shaded. Calculations of the probabilities for the simple geometric elements are described in detail in [2, 3]. In this case we get (Fig.2, b):

$$N_1'(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta);\quad N_1''(\xi,\eta)=-\xi-\eta-1.$$

Now, according to the formula of multiplying the probabilities of two independent events we get (9).

Note. In designing the angular shape functions $N_i(\xi,\eta)\left(i=\overline{1,4}\right)$ in some models of higher order it is convenient to use a triangle with a curved hypotenuse.

When building $N_5(\xi,\eta)$, two simple FE are considered (Fig.2, c): 5-2-3-7 and 5-7-4-1. In this case, the formula of multiplication gives the probability $N_5(\xi,\eta)$. It is clear that functions of the quadratic form FE build $N_1(\xi,\eta)$ and $N_5(\xi,\eta)$. All functions are given in [5]. The results are summerized in the following theorem.

Theorem 1. On a quadratic FE serendipity family, there is only an 8-parametric polynomial, satisfying the conditions (2), (3).

3. A cubic finite element serendipity family.

Theorem 2. At the cubic FE serendipity family a unique 12-parametric interpolation polynomial that satisfies the conditions of type (2), (3).

Proof: this element is shown in Fig.3, a, b.

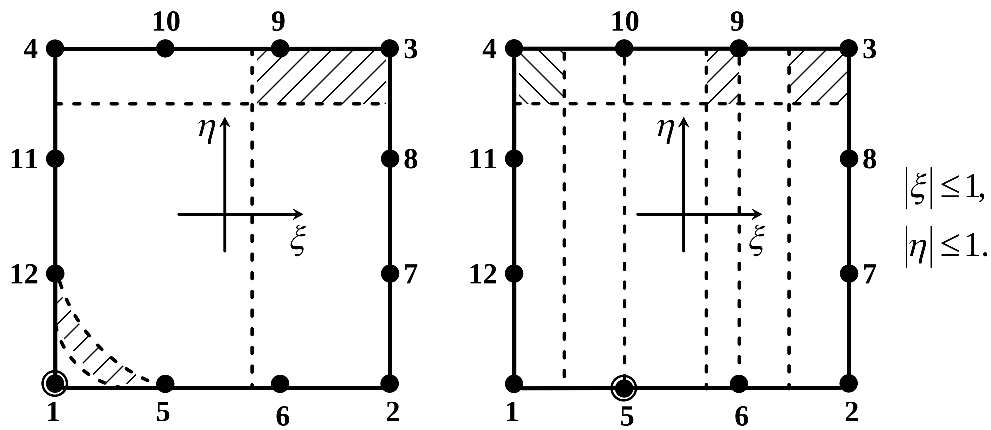

Fig.3 Cubic FE: (a) - sub-element composition to $N_1(\xi,\eta)$

(b) - sub-element composition to $N_5(\xi,\eta)$

A cubic function changes $N_1(\xi,\eta)$ along the borders of 1-2 and 1-4, and can be obtained by using two FEs: 1-2-3-4 linear and the quadratic 1-5-12 (Fig.3, a). The right triangle with a curved hypotenuse is independent [7]. Here it is used for the construction of $N_1(\xi,\eta)$ on the cubic FE. As the hypotenuse of 1-5-12 is a circle passing through the nodes 5-6-7-8-9-10-11-12. First of all, apply the sub-element 1-2-3-4 and determine the likelihood of a random point in the rectangle opposite node 1 (shaded). Then

$$p_1 = \frac{1}{4}(1-\xi)(1-\eta).$$

In the sub-element 1-5-12, the probability of an accidental fall into the strip region $\frac{10}{9} \le \xi^2 + \eta^2 \le 2$ and is

$$p_2 = \frac{\pi\left(\xi^2+\eta^2\right) - \pi\cdot\frac{10}{9}}{2\pi - \frac{10}{9}\cdot\pi} = \frac{1}{8}\left(9\left(\xi^2+\eta^2\right)-10\right)$$

Now, according to the formula of multiplying probabilities we get:

$$N_1(\xi,\eta) = \frac{1}{32}(1-\xi)(1-\eta)\left(9\left(\xi^2+\eta^2\right)-10\right) \qquad (10)$$

Generally, for $i = 1,2,3,4$

$$N_i(\xi,\eta) = \frac{1}{32}(1+\xi_i\xi)(1+\eta_i\eta)\left(9\left(\xi^2+\eta^2\right)-10\right), \quad \xi_i, \eta_i = \pm 1 \qquad (11)$$

To construct $N_5(\xi,\eta)$ (Fig.3, b) the three-FE with the general node 5: 5-2-3-10, 5-6-9-10 and 5-10-4-1 is considered. The area of "successful" outcomes is shaded. It is clear that we had thrown three random points (one in each FE).The probability of falling into a joint hatched rectangles is given by

$$N_5(\xi,\eta)=\frac{9}{32}(1-\eta)\left(1-\xi^2\right)(1-3\xi)$$

Generally, for units 5, 6, 9, 10:

$$N_i(\xi,\eta)=\frac{9}{32}(1+\eta_i\eta)\left(1-\xi^2\right)(1+9\xi_i\xi),\ \eta_i=\pm1;\ \xi_i=\pm\frac{1}{3} \tag{12}$$

For nodes 7, 8, 11, 12:

$$N_i(\xi,\eta)=\frac{9}{32}(1+\xi_i\xi)\left(1-\eta^2\right)(1+9\eta_i\eta),\ \eta_i=\pm\frac{1}{3};\ \xi_i=\pm1 \tag{13}$$

These functions form are known to be in the FEM [1, 4-7].The method of the inverse matrix, the procedure of Taylor, PGM provide a convergence of results.

Note. In this case, to the polynomial (4), the components $\alpha_9\xi^3$, $\alpha_{10}\eta^3$, $\alpha_{11}\xi^3\eta$, $\alpha_{12}\xi\eta^3$ are added. This polynomial is called the standard polynomial [6]. The main disadvantages of the standard model FEST-12: the existence of multiple zeros at the nodes, as well the unnatural uniform distribution of mass in units of force (negative pressure in the corner nodes).

These shortcomings can be overcome by PGM. It is a new method can that detect the hidden parameters of the interpolation, which can alter the functions of the form of quantitative and qualitative. From Pascal's triangle, it is easy to establish that the pattern of behavior $N_i(\xi,\eta)$ on the appropriate side FESF-12 allows for variation in the number of parameters of a polynomial of 12 to 16.

Theorem 3. At the cubic FE serendipity family, there is a 13-parametric interpolation polynomial that satisfies the conditions of type (2), (3).

Proof : we will proceed with a constructive manner. To do this, consider a different composition of simple sub-element to FESF-12 (Fig. 4, a, b).

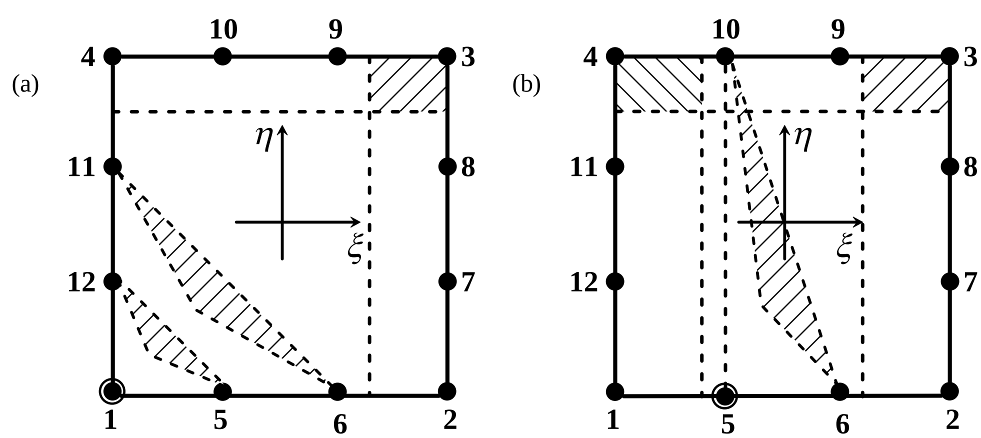

Fig.4 Cubic FE: (a) - sub-element composition to build; $N_1(\xi,\eta)$
(b) - sub-element composition to build $N_5(\xi,\eta)$

To construct $N_1(\xi,\eta)$, three simple sub-element (Fig. 4, a) with a common node 1: 1-2-3-4, 1-5-12, 1-6-11 are considered. In each sub-element, thrown random point subdomain "successful" outcomes of the tests are shaded. The probability of falling within the shaded area is expressed in terms of the relative area of sub-"successful" outcomes:

$$p_1 = \frac{1}{4}(1-\xi)(1-\eta); \quad p_2 = \frac{1}{2}(-3\xi - 3\eta - 4); \quad p_3 = \frac{1}{4}(-3\xi - 3\eta - 2).$$

According to the rule of multiplying the probabilities we get:

$$N_1(\xi,\eta) = \frac{1}{32}(1-\xi)(1-\eta)\left(9(1+\xi+\eta)^2 - 1\right).$$

Generally, for $i = 1, 2, 3, 4$

$$N_i(\xi,\eta) = \frac{1}{32}(1+\xi_i\xi)(1+\eta_i\eta)\left(9(1-\xi_i\xi-\eta_i\eta)^2 - 1\right), \quad \xi_i, \eta_i = \pm 1 \tag{14}$$

To construct $N_5(\xi,\eta)$, three sub-elements (Fig. 4, b) with a common node 5: 5-2-3-10, 5-10-4-1,and 5-6-10 are considered. Subdomain "successful" outcomes are shaded. The product of the probability of falling into the subdomain "successful" outcomes provides:

$$N_5(\xi,\eta) = \frac{9}{32}\left(1-\xi^2\right)(1-\eta)(-3\xi-\eta)$$

Generally, for $i = 5, 6, 9, 10$

$$N_i(\xi,\eta) = \frac{9}{32}\left(1-\xi^2\right)(1+\eta_i\eta)(9\xi_i\xi+\eta_1\eta), \quad \xi_i = \pm\frac{1}{3}; \ \eta_i = \pm 1 \tag{15}$$

For nodes $i = 7, 8, 11, 12$

$$N_i(\xi,\eta) = \frac{9}{32}\left(1-\eta^2\right)(1+\xi_i\xi)(\xi_i\xi+9\eta_1\eta), \quad \xi_i = \pm 1; \ \eta_i = \pm\frac{1}{3} \tag{16}$$

Testing shows that the conditions $N_i(\xi_k,\eta_k) = \delta_{ik}, \sum_{i=1}^{12} N_i(\xi,\eta) = 1$ are met, but the interpolation polynomial

$$P(\xi,\eta) = \sum_{i=1}^{12} N_i(\xi,\eta)\cdot f_i$$

has the 13th parameter $\alpha_{13}\xi^2\eta^2$. The modified model (14) - (16) differs from the standard model (11) - (13) so that the number of multiple zeros is decreased, and the negative pressure in the uniform distribution of mass forces on the nodes disappeared.

Corollary. The emergence of alternative bases (14) - (16) allows to generate a myriad of bases on a formula weighted averaging:

$$\overline{N}_i(\xi,\eta) = \alpha\cdot N_i^S(\xi,\eta) + (1-\alpha)N_i^M(\xi,\eta) \tag{17}$$

where N^S - the standard basis; N^M - Modified base; $0 \le \alpha \le 1$- Weighting factor.

Conclusion. The emergence of alternative bases for the elements of serendipity family opens up new possibilities for solving the problem of optimizing the properties of serendipity models. The proof of the existence of FESF-12 interpolation polynomials 14, 15 and 16 parameters is of great interest. In addition, PGM offers great promise for generating functions of the form of serendipity elements in 3D.

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