Analysis of Non-Newtonian Unsteady Thin Film MHD Flow on a Vertical Moving and Oscillating Belt

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ABSTRACT

Magnetohydrodynamic (MHD) thin film flow of an unsteady third order fluid is considered. The non-linear partial differential equations are solved analytically by using Optimal Homotopy Asymptotic Method (OHAM) and Adomian Decomposition Method (ADM). The Comparison of both methods is analyzed numerically and graphically. The effects of model parameters have also been studied.

KEYWORDS: Unsteady thin film flows, MHD, Lifting, Drainage, Third order fluid, OHAM and ADM.

I INTRODUCTION

Non-Newtonian thin film flows have large applications in a number of technological processes including production of polymer films or thin sheets. In physical configuration of non-Newtonian fluids it is complicated to clarify their mechanical manners by a particular constitutive equation. For this reason, a great variety of constitutive equations have been proposed \cite{1}. Siddiqui et al. \cite{2} studied the thin film flow of Sisko fluid and Oldroyd 6-constant fluid on a vertical moving belt. The nonlinear equations governing the flow solved using homotopy perturbation method. Volume flux and average velocity are also calculated. Hayat and Sajid \cite{3} investigated the comparison between Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM) for thin film flow of non-Newtonian fluids on moving belt. Nargis and Tahir \cite{4} investigated the thin film flow of a third order fluid in two cases when the fluid moves down an inclined plane and moves on a moving belt. The volume flux and average film velocity are also discussed.

There is no single constitutive equation that can be used to examine all the non-Newtonian fluids, different linear and non-linear equations have been proposed. A third grade fluid is a subclass of non-Newtonian fluid and its governing non-linear equation has successfully studied and treated in many literatures. Ariel \cite{5} discussed the steady and laminar flow of a third grade fluid through a porous flat channel. The flow is governed by a non-linear boundary value problem and different numerical methods are developed to obtain the appropriate solution.

Sahoo et al. \cite{6} investigated the non-Newtonian boundary layer flow and heat transfer over an exponentially stretching sheet with uniform transverse magnetic field. They find the combined effects of the partial slip and the third grade fluid parameters on the velocity profile. Islam et al.\cite{7, 8} discussed the unsteady second grade fluid flow between wire and die. The problem is solved by OHAM and the ideas of OHAM extend not for the solution of linear and non-linear differential equations but also can be applied for linear and non-linear partial differential equations.

The study of unsteady magnetohydrodynamics (MHD) thin film flows has received substantial concentration in the past due to its applications in the field of engineering, polymer industry and petroleum industries. TazaGul et al. \cite{9} investigated the heat transfer analysis in electrically conducting third grade thin film fluid. They studied the combined effect of heat and MHD on the velocity field and the effects of model parameters on velocity, skin friction and temperature variation. Khan et al. \cite{10-12} discussed the solution of the unsteady flow of an incompressible, electrically conducting third grade fluid bounded by porous plate using homotopy analysis methods (HAM). The analytical solutions are shows through graph. Ali et al. \cite{13} investigated the numerical solution of electrically conducting fluid flow and heat transfer over porous stretching sheet. The governing non-linear partial differential equations of motion have been numerically solved by Method of Stretching Variables. The effects of physical parameters Magnetic parameter, Grashof number, Prandlt number and injection parameter S have been observed on velocity, temperature distributions. Idrees et al. \cite{14} studied the low of incompressible fluid between two parallel plates and the governing fourth order non linear differential equation is solved by using Optimal Homotopy Asymptotic Method. This method is effective, sampler and easier. Yongqi and Wu \cite{15} discussed the unsteady flow of an incompressible fourth grade fluid in a uniform magnetic field and the unsteady flow is induced by oscillating two-dimensional infinite porous plate. They compared the flow behavior of the fourth-grade non-Newtonian fluid with the Newtonian fluid. Aiyesimi et al. \cite{16} investigated thin film flow of an MHD third grade fluid down an inclined plane. The solutions of
problem obtained by traditional perturbation and homotopy perturbation technique. The effect of slip parameter, Magnetic parameter and other parameters involved in the problem are discussed. Alam et al. [17] discussed the magnetohydrodynamic (MHD) thin-film flow of the Johnson–Segalman fluid for lifting and drainage problems on vertical plane. The nonlinear differential equations are solved analytically using the Adomian decomposition method (ADM) and discuss the effect of different parameters on velocity field. Ajadi [18] investigated the viscoelastic fluid between oscillatory walls in the presence of magnetic field and obtained the closed form solutions for velocity and temperature profiles. The shear stress and Nusselt number are found at the walls and show the results graphically. Liao [19] investigated the homotopy analysis method, for nonlinear problems which give the analytic solutions of magnetohydrodynamic viscous flows of non-Newtonian fluids over a stretching sheet. He expresses the solution of nonlinear differential equations of second order and third order power law fluids. Gamal [20] examined the effects of magnetohydrodynamics (MHD) on thin film of unsteady micro polar fluid through a porous medium. He considered these thin films for three unlike geometries and the governing continuity, momentum and angular momentum equations are converted into a system of nonlinear ordinary differential equations.

The main aim of the current work is to the study of thin film of a third grade fluid on a vertical oscillating belt under the influence of magnetohydrodynamics (MHD) using adomian decomposition method and optimal Homotopy Asymptotic Method. In 1992 Adomian [21, 22] introduced the ADM for the approximate solutions for linear and non linear problems. Wazwaz [23, 24] used ADM for the reliable treatment of Bratu-type and Rmden-Fowler equations. Idrees et al. [25] studied the Optimal Homotopy Asymptotic Method (OHAM). They apply OHAM on wave equations in different forms and obtained the solution. They show that OHAM is successful, simpler and easier method. Duan [25] investigated the solution of multi-order and multi-point boundary value problems for nonlinear ordinary differential equations and partial differential equations by using the Adomian decomposition method (ADM). They use Duan’s convergence parameter, which provides a significant advantage during the calculations of the solution components for nonlinear boundary value problems. Husain and Ahmad et al. [26-27] discussed different numerical techniques for the combined effect of MHD and porosity. They have shown the effect of different parameters on the velocity as well as on pressure field.

II BASIC EQUATION AND FORMULATION OF THE FIRST PROBLEM

We assume an oscillating and vertically upward moving belt. \( V \) is the uniform velocity of the belt. A uniform Magnetic field is applied transversely to the belt. The belt carries with itself a thin layer of third order liquid during it upward motion. The thickness of the fluid is uniform and considered as uniform. By using the above assumptions and Equations (1) the continuity equation (3) is satisfied identically and the boundary value problems for nonlinear ordinary differential equations are converted into a system of non-linear homogeneous.

The continuity equation is

\[
\nabla \cdot \mathbf{v} = 0.
\]

By using the above assumptions and Equations (1) the continuity equation (3) is satisfied identically and the momentum equation reduces to the form

\[
\rho \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial}{\partial x} \left( \mu \frac{\partial \mathbf{v}}{\partial x} \right) + \rho g - \sigma B_0^2 \mathbf{v},
\]

The Cauchy stress component \( T_{xy} \) of the third order fluid is

\[
T_{xy} = \mu \frac{\partial \mathbf{v}}{\partial x} + \alpha_1 \frac{\partial}{\partial t} \left( \frac{\partial \mathbf{v}}{\partial x} \right) + \beta_3 \frac{\partial^2 \mathbf{v}}{\partial t^2} + 2(\beta_1 + \beta_3) \left( \frac{\partial \mathbf{v}}{\partial x} \right)^2 = \lambda_3 \mathbf{v}.
\]

Putting equation (5) in (4) we get

\[
\rho \frac{\partial \mathbf{v}}{\partial t} = \mu \frac{\partial^2 \mathbf{v}}{\partial x^2} + \rho \alpha_1 \frac{\partial}{\partial t} \left( \frac{\partial \mathbf{v}}{\partial x} \right) + 6\beta_3 \left( \frac{\partial \mathbf{v}}{\partial x} \right)^2 \frac{\partial \mathbf{v}}{\partial t} - \rho g - \sigma B_0 \mathbf{v}.
\]

Introducing non-dimensional variables as

\[
\hat{\nu} = \frac{\nu}{V}, \quad \hat{x} = \frac{x}{\sigma}, \quad \hat{t} = \frac{t\nu}{\rho \sigma^2}, \quad \alpha = \frac{\alpha_1 \nu}{\rho \sigma^2}, \quad \beta = \frac{\beta \nu^2}{\rho \sigma^2}, \quad \lambda_3 = 6 \beta \left( \frac{\partial \nu}{\partial x} \right)^2 \frac{\partial \nu}{\partial t} - S_1 - M V,
\]

Where \( M \) is the magnetic parameter, \( \beta \) is the non-Newtonian parameter, \( S_1 \) is the stock number and \( \alpha \) is the non-dimensional variable.

Using the above dimensionless variables in equation (15) and dropping bars we obtain

\[
\frac{\partial \hat{\nu}}{\partial t} = \frac{\partial^2 \hat{\nu}}{\partial x^2} + \frac{\partial}{\partial t} \left( \frac{\partial \hat{\nu}}{\partial x} \right) + 6\beta \left( \frac{\partial \hat{\nu}}{\partial x} \right)^2 \hat{\nu} - S_1 - \hat{\nu} V,
\]

And the boundary conditions are
\[ v(a, t) = 1 + \cos \omega t, \quad \frac{\partial v(t)}{\partial x} = 0, \]  

(9)

\[ v(x, t) = \sum_{k=0}^{\infty} v_k(x, t) \]  

where \( \sum_{k=0}^{\infty} v_k(x, t) \) is the Adomian polynomial and equation (18) becomes

\[ L_x v(x, t) = \Psi - L_x^{-1} \mathcal{R} \left( \sum_{n=0}^{\infty} v_n(x, t) \right) - L_x^{-1} \mathcal{M} \left( \sum_{n=0}^{\infty} v_n(x, t) \right), \]  

(12)

\[ L_x^{-1} = \int dx dx, \]  

(13)

\[ L_x^{-1} L_x v(x, t) = L_x^{-1} \Psi - L_x^{-1} \mathcal{R} \left( \sum_{n=0}^{\infty} v_n(x, t) \right) - L_x^{-1} \mathcal{M} \left( \sum_{n=0}^{\infty} v_n(x, t) \right), \]  

(14)

\[ v(x, t) = L_x^{-1} \left( \Psi - L_x^{-1} \mathcal{R} \left( \sum_{n=0}^{\infty} v_n(x, t) \right) - L_x^{-1} \mathcal{M} \left( \sum_{n=0}^{\infty} v_n(x, t) \right) \right), \]  

(15)

where \( \Psi \) represent the term produced from the integration of the \( \mathcal{R} \) and using the boundary conditions. Since Adomian Decomposition Method (ADM) is a series solution method, then result of \( v(x, t) \) can be given as

\[ v(x, t) = \sum_{n=0}^{\infty} v_n(x, t), \]  

(17)

From equation (1.20)

\[ \sum_{n=0}^{\infty} v_n(x, t) = \Psi - L_x^{-1} \mathcal{R} \left( \sum_{n=0}^{\infty} v_n(x, t) \right) - L_x^{-1} \mathcal{M} \left( \sum_{n=0}^{\infty} v_n(x, t) \right), \]  

(18)

\[ \sum_{n=0}^{\infty} A_n = A_n, \]  

(19)

\[ M \left( \sum_{n=0}^{\infty} v_n(x, t) \right) = \sum_{n=0}^{\infty} A_n, \]  

(20)

Write equation (20) in components form

\[ v_0(x, t) + v_1(x, t) + v_2(x, t) = \Psi - L_x^{-1} \mathcal{R} \left( v_0(x, t) + v_1(x, t) + v_2(x, t) \right) - L_x^{-1} (A_0 + A_1 + A_2 \ldots), \]  

(21)

Comparison for different components of velocity profile \( v_0(x, t) + v_1(x, t) + v_2(x, t) \ldots \), where the function \( \Psi \) described the zero component \( v_0(x, t) \).

\[ v_0(x, t) = \Psi \]  

\[ v_1(x, t) = -L_x^{-1} [R v_0(x, t)] - L_x^{-1} [A_0] \]  

\[ v_2(x, t) = -L_x^{-1} [R v_1(x, t)] - L_x^{-1} [A_1] \]  

\[ v_3(x, t) = -L_x^{-1} [R v_2(x, t)] - L_x^{-1} [A_2] \]  

\[ \vdots \]  

\[ v_n(x, t) = -L_x^{-1} [R v_{n-1}(x, t)] - L_x^{-1} [A_n] \]  

(22)
IV OPTIMAL HOMOTOPY ASYMPTOTIC METHOD

Here we discussed the fundamental introduce of OHAM. In order to explain OHAM, we consider the following boundary value problem

\[ \mathcal{L}(v(x, t)) + \mathcal{N}(v(x, t)) + \mathcal{S}(v(x, t)) = 0, \quad \mathcal{B}(v(x, t)) = 0, \quad x \in \Gamma, \quad (24) \]

Where \( \mathcal{L} \), is the linear operator in differential equation, \( v(x, t) \) is unknown function, \( \mathcal{N} \) is non linear term, \( \mathcal{S} \) is source term, \( x \) is spatial variable and \( t \) is time variable, \( \Gamma \) is domain of problem and \( \mathcal{B} \) is boundary operator. Equation (24) is called optimal homotopy equation.

According to OHAM, we put the optimal homotopy

\[ \Psi(x, t, \rho); \Gamma \times [0, 1] \rightarrow \mathbb{R}, \quad (25) \]

Satisfying the following equation

\[ [1 - \rho]\mathcal{H}[\Psi(x, t, \rho) + \mathcal{S}(x, t)] - \mathcal{H}(\rho) [\mathcal{L}(\Psi(x, t, \rho) + \mathcal{N}(x, t, \rho)) + \mathcal{S}(x, t)] = 0, \quad (26) \]

Where \( \rho \) is embedding parameter and \( \rho \in [0, 1] \), \( \mathcal{H}(\rho) \) is the auxiliary function which is defined as

\[ \mathcal{H}(\rho) = \rho \mathcal{C}_1 + \rho^2 \mathcal{C}_2 + \rho^3 \mathcal{C}_3 \ldots, \quad (27) \]

\( \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \) are called auxiliary constants which is find latter.

From equation (26) obviously we can write

\[ \rho = 0 \Rightarrow \mathcal{H}[\Psi(x, t, 0), 0] = \mathcal{L}(\Psi(x, t, 0) + \mathcal{S}(x, t) = 0, \quad (28) \]

\[ \rho = 1 \Rightarrow \mathcal{H}[\Psi(x, t, 1), 1] = \mathcal{H}(1)[\mathcal{L}(\Psi(x, t, 1) + \mathcal{S}(x, t) + \mathcal{N}(x, t, \rho))] = 0, \quad (29) \]

Here in equation (26), \( \Psi(x, t, \rho) \) is unknown function and clearly it holds that when \( \rho = 0 \) then \( \Psi(x, t, 0) = v_0(x, t) \) and when \( \rho = 1 \) then \( \Psi(x, t, 1) = v(x, t) \)

Generally we can write

\[ \Psi(x, t, \rho, \mathcal{C}_i) = v_0(x, t) + \sum_{k=1}^{\infty} v_k(x, t, \mathcal{C}_i) \rho^k, \quad i = 1, 2, 3, \ldots, m. \quad (30) \]

To find the component of unknown function \( v(x, t) \), we Substitute equation (30) into equation (26) collecting the same power of \( \rho \) and putting each coefficient of \( \rho \) equal to zero. So the differential equation is solved by using the given boundary conditions, we get the value of \( v_0(x, t), v_1(x, t, \mathcal{C}_i) \text{and} v_2(x, t, \mathcal{C}_2) \text{as} \)

First order problem

\[ \mathcal{L}(v_1(x, t)) = \mathcal{C}_1 \mathcal{N}_0(v_0(x, t)), \quad \mathcal{B}(v_1(x, t)) = 0, \quad (31) \]

Second order problem

\[ \mathcal{L}(v_2(x, t)) - \mathcal{L}(v_1(x, t)) = \mathcal{C}_2 \mathcal{N}_0(v_0(x, t)) + \mathcal{C}_1 \mathcal{N}(v_1(x, t), v_1(x, t)), \quad \mathcal{B}(v_2(x, t)) = 0. \quad (32) \]

Similarly kth order problem

\[ \mathcal{L}(v_k(x, t)) - \mathcal{L}(v_{k-1}(x, t)) = \mathcal{C}_k \mathcal{N}_0(v_0(x, t)) + \sum_{i=1}^{k-1} \mathcal{C}_i \mathcal{N}(v_{k-i}(x, t), v_{k-1}(x, t), v_k(x, t)), \quad \mathcal{B}(v_k(x, t)) = 0, \quad k = 2, 3, 4 \ldots, (33) \]

The general solution of equation (26) is of the form

\[ v^n = v_0(x, t) + \sum_{k=1}^{\infty} v_k(x, t, \mathcal{C}_i), \quad (34) \]

By combining equations (26) and (34) we obtained the residual as

\[ \mathcal{R}(x, \mathcal{C}_1) = \mathcal{L}(v(x, t, \mathcal{C}_1)) + \mathcal{S}(x, t) + \mathcal{N}(v(x, t, \mathcal{C}_1)), \quad (35) \]

Several methods are used to find the optimal value of auxiliary constants \( \mathcal{C}_i, i = 1, 2, 3 \ldots, \text{that is Galerkin’s Method, Ritz Method, Least Squares Method and Collocation Method. Here in the present problem we introduced the Method of Least Squares, given by} \]

\[ \mathcal{J}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_i) = \int_a^b \mathcal{R}^2(x, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_i) dx, \quad (36) \]

The value of \( a \) and \( b \) are depending on the domain of problem. The auxiliary constants \( \mathcal{C}_i, i = 1, 2, 3, \ldots, n \) can be identified from the given conditions

\[ \frac{\partial v_1}{\partial \mathcal{C}_1} = \frac{\partial v_2}{\partial \mathcal{C}_2} = \frac{\partial v_3}{\partial \mathcal{C}_3} = \ldots = \frac{\partial v_n}{\partial \mathcal{C}_n} = 0, \quad (37) \]

The auxiliary constants \( \mathcal{C}_i \) can also be finding from the other methods. At last by using the values of auxiliary constants \( \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_i \), the approximate solution is well determined.

V THE ADM SOLUTION OF FIRST PROBLEM

In operator form equation (8) can be written as

\[ L_x v(x, t) = \frac{\partial v}{\partial t} - \alpha \left( \frac{\partial^2 v}{\partial x^2} \right) - 66 \left( \frac{\partial v}{\partial x} \right)^2 \left( \frac{\partial^2 v}{\partial x^2} \right) + S_t + M_v, \quad (38) \]

Using the inverse operator, \( L_x^{-1} \) on equation (38) we get

\[ L_x^{-1} L_x v(x, t) = L_x^{-1} \left( \frac{\partial v}{\partial t} + S_t \right) - \alpha L_x^{-1} \left( \frac{\partial^2 v}{\partial x^2} \right) - 66 \left( \frac{\partial v}{\partial x} \right)^2 \left( \frac{\partial^2 v}{\partial x^2} \right) + ML_x^{-1} v, \quad (39) \]

\[ v(x, t) = \Psi - \alpha L_x^{-1} \left( \frac{\partial^2 v}{\partial x^2} \right) - 66 \left( \frac{\partial v}{\partial x} \right)^2 \left( \frac{\partial^2 v}{\partial x^2} \right) + ML_x^{-1} v. \quad (40) \]

Since ADM is a series method then the solution \( v \) and nonlinear term can be expressed as

\[ v(x, t) = \sum_{n=0}^{\infty} v_n(x, t), \quad (41) \]

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Nasir et al., 2014
The value of $g_3105/g_2870$
First component problem:
Solution of zero component problem using boundary condition in equation (9) is

\[
\sum_{n=0}^{\infty} A_n, \quad \frac{d}{dt} \left( \frac{\partial^2 v_0}{\partial x^2} \right) = \sum_{n=0}^{\infty} B_n, \quad \sum_{n=0}^{\infty} v_n(x, t) = \Psi - \alpha L_x^{-1} \left( \sum_{n=0}^{\infty} A_n \right) - 6\beta L_x^{-1} \left( \sum_{n=0}^{\infty} B_n \right) + ML_x^{-1} \sum_{n=0}^{\infty} v_n(x, t),
\]

From equation (43) the adomian polynomials $A_n$ and $B_n$ in component from are given

\[
A_0 = \frac{d}{dt} \left( \frac{\partial^2 v_0}{\partial x^2} \right), \quad B_0 = \left( \frac{\partial v_0}{\partial x} \right)^2,
\]

\[
A_1 = \frac{d}{dt} \left( \frac{\partial^2 v_1}{\partial x^2} \right), \quad B_1 = 2 \left( \frac{\partial v_0}{\partial x} \right) \left( \frac{\partial^2 v_1}{\partial x^2} \right) + \left( \frac{\partial v_0}{\partial x} \right)^2, \quad v_0(x, t) + v_1(x, t) + \cdots = g(x, t) - \alpha L_x^{-1} (A_0 + A_1 + \cdots) - 6\beta L_x^{-1} (B_0 + B_1 + \cdots) + M(v_0(x, t) + v_1(x, t) + \cdots),
\]

The components of velocity profile are obtained by comparing both side of equation (47)

**Zero component problem:**

\[
v_0(x, t) = L_x^{-1} S_x,
\]

Solution of zero component problem using boundary conditions given in equation (9) is:

\[
v_0(x, t) = \left( 1 + \cos[t \omega] \right) - \left( 1 + \cos[t \omega] \right) x + \left( \frac{\omega}{2} \right)^2 x^2.
\]

**First component problem:**

\[
v_1(x, t) = ML_x^{-1} v_0 + L_x^{-1} \left( \frac{\partial v_0}{\partial x} \right) - \alpha L_x^{-1} (A_0) - 6\beta L_x^{-1} (B_0),
\]

Solution of first component problem using boundary conditions given in equations (9) is:

\[
v_1(x, t) = \frac{M}{4} \left( \frac{\partial \omega}{\partial x} \right)^2 x + \left( \frac{M}{3} \right) \left( 1 + \frac{1}{2} \cos[t \omega] + \frac{1}{2} \cos[2t \omega] \right) - \frac{1}{2} \omega \sin[t \omega] - 3\beta S_t \left( 1 + \frac{1}{2} \cos[t \omega] + \frac{1}{2} \cos[2t \omega] \right), \quad \beta S_t^2 \left( 1 + \cos[t \omega] \right) - \frac{\omega}{2} \alpha \left( \frac{\partial v_0}{\partial x} \right)^2 + \frac{\omega}{2} \frac{\partial^2 v_0}{\partial x^2} + \left( \frac{\omega}{2} \right)^2 \left( 1 + \cos[t \omega] \right) x + \left( \frac{\omega}{2} \right)^2 x^2.
\]

**Second component problem:**

\[
v_2(x, t) = ML_x^{-1} v_1 + L_x^{-1} \left( \frac{\partial v_1}{\partial x} \right) - \alpha L_x^{-1} (A_1) - 6\beta L_x^{-1} (B_1),
\]

The solution of second component of velocity distribution using boundary conditions in equations (9) is too large. So derivation are given up to first order while graphical solutions are given up to second order.

**VI THE OHAM SOLUTION OF FIRST PROBLEM**

We write equation (8) in standard form of OHAM and study zero, first and second component problems

**Zero component problem:**

\[
\frac{\partial^2 v_0(x, t)}{\partial x^2} = S_t,
\]

Solution of zero component problem using boundary condition in equation (9) is:

\[
v_0(x, t) = \left( 1 + \cos[t \omega] \right) - \left( 1 + \cos[t \omega] \right) x + \left( \frac{\omega}{2} \right)^2 x^2.
\]

**First component problem:**

\[
\frac{\partial^2 v_1(x, t)}{\partial x^2} = \beta S_t \left( 1 + \cos[t \omega] \right) - \omega \cos[t \omega] + \frac{\partial v_0}{\partial x} + c_1 \left( \frac{\partial v_0}{\partial x} \right)^2 + c_1 \left( \frac{\partial^2 v_0}{\partial x^2} \right) + 6\beta \left( \frac{\partial v_0}{\partial x} \right)^2 \left( \frac{\partial^2 v_0}{\partial x^2} \right) + \alpha \frac{\partial^4 v_0}{\partial x^4},
\]

Solution of first component problem using boundary condition in equation (9) is:

\[
v_1(x, t) = \left[ 2 \cos[t \omega] \left( \frac{M}{3} - \beta S_t^2 \right) - \omega \cos[t \omega] \sin[t \omega] \left( \frac{\omega}{2} \right) - 3\beta \cos[t \omega] S_t - \frac{3}{2} \beta \cos[2t \omega] S_t - \frac{1}{4} \beta S_t^2 \right] x + \left[ \frac{9}{2} \beta S_t - \frac{3}{2} \beta S_t \left( 1 + \alpha \right) \cos[t \omega] S_t + \frac{3}{2} \beta \cos[2t \omega] S_t + 6\beta \cos[t \omega] \sin[t \omega] + \frac{M S_t}{12} - \frac{1}{3} \omega \cos[t \omega] \sin[t \omega] + \frac{M S_t}{12} - 4\beta \cos[t \omega] \sin[t \omega] \times \sin[t \omega] + c_1 \beta S_t^3 \right] x^3 + \left[ \frac{M S_t}{12} - \frac{M S_t}{24} \right] x^4.
\]

The solution of second component of velocity distribution is too large. So derivation are given up to first order while graphical solutions are given up to second order.

The optimal value of $c_j = 1, 2, 3 \ldots$ are find by using the residual

\[
R = f(v(x, t, c_j)) + T(v(x, t, c_j)) + S(v(x, t, c_j)),
\]

The value of $c_j$ for the velocity components $v_0(x, t), v_1(x, t, c_0)$ and $v_2(x, t, c_2)$ are

\[
c_1 = -0.0481075191, \quad c_2 = -0.0082765719
\]
VII FORMULATION OF THE SECOND PROBLEM

In this problem the belt is only oscillating and not moving upward. The remaining assumptions are same as in previous problem. The fluid layer is draining down the belt due to gravity. Therefore, the stock number in equation (8) positively mentioned.

Boundary condition for electrically conducting drainage problem is:

\[ v(0, t) = V \cos(\omega t), \quad \frac{\partial v(0, t)}{\partial x} = 0, \quad (48) \]

Due to downward flow of fluid film equation (8) reduced as.

\[ \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \alpha \frac{\partial v}{\partial t} \left( \frac{\partial^2 v}{\partial x^2} \right) + 6\beta \left( \frac{\partial v}{\partial x} \right)^2 + S_x - M v, \quad (49) \]

Boundary conditions are

\[ v(0, t) = \cos(\omega t), \quad \frac{\partial v(1, t)}{\partial x} = 0, \quad (50) \]

VIII THE ADM SOLUTION OF SECOND PROBLEM

Using adomian decomposition method (ADM) on equation (49). The adomian polynomials (34-36) are same for both lifting and drainage problems. The velocity components are

Zero component problem:

\[ v_0(x, t) = \Psi = -x^{-1} S_t, \quad (51) \]

Solution of Zero component problem using boundary conditions in equation (50):

\[ v_0(x, t) = \cos[t\omega] - \left( \cos[t\omega] + \frac{S_t}{2} \right) x - \frac{S_t}{2} x^2, \quad (52) \]

First component problem:

\[ v_1(x, t) = M L_x^{-1} v_0 + L_x^{-1} \left( \frac{\partial v_0}{\partial t} \right) - \alpha L_x^{-1} (A_0) - 6\beta L_x^{-1} (B_0), \quad (53) \]

Solution of first component drainage problem using boundary condition in equation (50):

\[ v_1(x, t) = \left[ \frac{1}{3} \omega \sin[t\omega] - \frac{1}{2} \beta \cos[2t\omega] S_t + \cos[t\omega] \left( \beta S_t^2 - \frac{M^2}{3} \right) - \frac{M S_t}{24} - \frac{3\beta S_t^2}{2} - \frac{\beta S_t^2}{4} \right] x + \left[ \cos[t\omega] \left( \frac{M}{2} - 3\beta S_t^2 \right) - \frac{3}{2} \beta \cos[2t\omega] S_t + \frac{3\beta S_t^2}{2} + \frac{3\beta S_t^2}{4} \right] x^2 + \left[ \frac{1}{6} \omega \sin[t\omega] + \frac{M S_t}{12} + \cos[t\omega] \left( 2\beta S_t^2 - \frac{M}{6} \right) - \beta S_t^2 \right] x^3 + \left[ \frac{\beta S_t^2}{2} - \frac{M S_t}{24} \right] x^4, \quad (54) \]

The solution of second component of velocity distribution using boundary conditions in equation (50) is too large. So derivation are given up to first order while, graphical solutions are given up to second order.

IX THE OHAM SOLUTION OF SECOND PROBLEM

We write equation (49) in standard form of OHAM and study zero, first and second component problems

Zero component problem:

\[ \frac{\partial^2 v_0(x, t)}{\partial x^2} = -S_t \quad (55) \]

Solution of zero component problem using boundary conditions in equation (50) is

\[ v_0(x, t) = \cos[t\omega] + \left[ \frac{S_t}{2} - \cos[t\omega] \right] x - \frac{S_t}{2} x^2, \quad (56) \]

First component problem:

\[ \frac{\partial^2 v_1(x, t)}{\partial x^2} = M c_1 v_0 - S_t - c_1 S_t + c_1 \left( \frac{\partial v_0}{\partial t} \right) + \alpha \left( \frac{\partial v_0}{\partial x} \right)^2 - \frac{\partial v_0}{\partial x} \left( \frac{\partial^2 v_0}{\partial x^2} \right) - 6\beta \left( \frac{\partial v_0}{\partial x} \right)^2 + \alpha \left( \frac{\partial^2 v_0}{\partial x^2} \right), \quad (57) \]

Solution of first component drainage problem using boundary condition in equation (50) is

\[ v_1(x, t) = c_1 \left[ \frac{1}{3} \omega \sin[t\omega] - \frac{1}{2} M \cos[t\omega] - 3\beta \cos[t\omega] S_t + \beta \cos[t\omega] S_t^2 - \frac{M S_t}{24} - \frac{3\beta S_t^2}{2} - \frac{\beta S_t^2}{4} \right] x + \left[ \cos[t\omega] \left( \frac{M}{2} - 3\beta S_t^2 \right) + \frac{3}{2} \beta \cos[2t\omega] S_t + \frac{3\beta S_t^2}{2} + \frac{3\beta S_t^2}{4} \right] x^2 + \left[ \cos[t\omega] \left( 2\beta S_t^2 - \frac{M}{6} \right) + \frac{M S_t}{12} + \beta S_t^2 \right] x^3 + \left[ \frac{\beta S_t^2}{2} - \frac{M S_t}{24} \right] x^4 \quad (58) \]

The solution of second component of velocity distribution is too large. So derivation are given up to first order while, graphical solutions are given up to second order.

The values of auxiliary constants for the drainage velocity profile are

\[ c_1 = 0.2664104869, \quad c_2 = -0.0766985178, \]
Table 1: Comparison of OHAM and ADM for the lift velocity profile, by taking, \( \omega = 0.2, \alpha = 0.02, S_t = 0.5, M = 0.2, \beta = 0.5, t = 20 \),

<table>
<thead>
<tr>
<th>x</th>
<th>OHAM</th>
<th>ADM</th>
<th>Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.3463563791</td>
<td>0.3463563791</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2898555823</td>
<td>0.2894507102</td>
<td>4.0487 \times 10^{-4}</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2381055645</td>
<td>0.2382704429</td>
<td>1.6487 \times 10^{-4}</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1911574650</td>
<td>0.1921874560</td>
<td>1.0299 \times 10^{-3}</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1490565622</td>
<td>0.1508437362</td>
<td>1.7907 \times 10^{-3}</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1118420991</td>
<td>0.1140937150</td>
<td>2.2516 \times 10^{-3}</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0795471302</td>
<td>0.0819022260</td>
<td>2.3550 \times 10^{-3}</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0521983909</td>
<td>0.0543240816</td>
<td>2.1256 \times 10^{-3}</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0298161893</td>
<td>0.0314392700</td>
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</tr>
<tr>
<td>0.9</td>
<td>0.0124143199</td>
<td>0.0133193725</td>
<td>9.0505 \times 10^{-4}</td>
</tr>
<tr>
<td>1.0</td>
<td>1.2567 \times 10^{-17}</td>
<td>-4.168 \times 10^{-15}</td>
<td>4.1808 \times 10^{-15}</td>
</tr>
</tbody>
</table>

Table 2: Comparison of OHAM and ADM for the drainage velocity profile, by taking, \( \omega = 0.2, \alpha = 0.02, S_t = 0.3, M = 0.2, \beta = 0.5, t = 10 \),

<table>
<thead>
<tr>
<th>x</th>
<th>OHAM</th>
<th>ADM</th>
<th>Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-0.41614683</td>
<td>-0.41614683</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
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<td>-0.36067480</td>
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<tr>
<td>0.2</td>
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<td>-0.30838045</td>
<td>1.2445 \times 10^{-3}</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.26077336</td>
<td>-0.25926007</td>
<td>1.5132 \times 10^{-3}</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.21485445</td>
<td>-0.21329333</td>
<td>1.5611 \times 10^{-3}</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.1785765</td>
<td>-0.17944089</td>
<td>1.4167 \times 10^{-3}</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.13176762</td>
<td>-0.13064261</td>
<td>1.1250 \times 10^{-3}</td>
</tr>
<tr>
<td>0.7</td>
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<td>-0.09381617</td>
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</tr>
<tr>
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</tr>
<tr>
<td>1.0</td>
<td>3.057 \times 10^{-17}</td>
<td>-8.72 \times 10^{-17}</td>
<td>1.1785 \times 10^{-16}</td>
</tr>
</tbody>
</table>

\( c_1 = -0.0481075191, c_2 = -0.0082765719, c_3 = 0.2664104869, c_4 = -0.0766985178 \)

Figure 3: Comparison of OHAM and ADM methods for lift velocity profile (on left) by taking \( \omega = 0.2, \alpha = 0.02, S_t = 0.5, M = 0.2, \beta = 0.5; t = 20 \) (on right) drainage velocity profile by taking \( \omega = 0.2, \alpha = 0.02, S_t = 0.3, M = 0.2, \beta = 0.5; t = 10 \).
RESULTS AND DISCUSSION

In the present work we study the non-Newtonian unsteady thin film MHD flow on a vertical moving and oscillating belt. From the model non-linear partial differential equations with boundary conditions are obtained and solved by using ADM and OHAM methods. The comparison of solutions and the graphical representations of the lifting and drainage velocities profile are shown. The effect of Magnetic parameter $M$, non-Newtonian parameter $\beta$, the stock number $S_t$ for both lift and drainage velocity profiles is discussed. In tables 1-2 the numerical comparison of OHAM and ADM methods at different time interval for lifting and drainage velocity profile are show and find the absolute error. From the tables we observe that the absolute error between ADM and OHAM are directly related to the time level. Fig 1-2 shows geometry of the problem. Fig. 3 shows the
comparison of ADM and OHAM solutions for both lifting and drainage velocity distribution by taking different values of the parameters. Fig. 4 are plotted in order to observe the influence of different time level on the velocity profile. Figs. 5-7 are plotted in order to see the effect of $M, S_l$, and $\beta$ on the lift and drainage velocity profile. Fig. 5 gives the effect of magnetic parameter $M$ on the lift and drainage velocity profile. The magnetic parameter $M$ has a direct relation for the lift velocity profile while the drainage velocity field decreases by increasing $M$. Fig. 6 shows the effect of stoke number $S_l$ on lift and drainage velocity field. The fig show that the lift velocity decreases when $S_l$ increases while in drainage case the stoke number $S_l$ have a direct variation for the velocity profile. It shows that increase in stock number causes the fluid motion due to its opposite direction. Fig. 7 gives the effect of Non-Newtonian parameter $\beta$ on the lift and drainage velocity. The fig show that the lift velocity profile decreases by increasing $\beta$ while in drainage case the Non-Newtonian parameter $\beta$ have a direct variation for the drainage velocity profile.

XI CONCLUSION

The unsteady thin film flow of an MHD third grade fluid on a vertical oscillating belt has been discussed. The constitutive equation governing the flow of a third grade fluid for lifting and drainage problems are solved analytically by using Adomian decomposition method and Optimal Homotopy Asymptotic Method. The numerical and graphical comparison of ADM and OHAM are discussed. The effect of stoke number $S_l$, non-Newtonian parameter $\beta$, magnetic parameter $M$ and other parameters involved in the problem are discussed and result are displayed in graph to observe the effect of these parameters on lifting and drainage velocity profile. It is concluded that velocity increases as the magnetic parameter $M$ increases in lifting case while velocity decreases as magnetic parameter increases in drainage case and velocity decreases as the non-Newtonian parameter increases in lifting case while in drainage case direct variation between them similarly the velocity decreases as the stoke number increases in lifting case while in drainage case direct variation between them.

REFERENCES


